4) Vectors

Where we study vectors:

We study vectors in $\mathbb{E}^3$

Euclidean 3-space

$\mathbb{E} = \{(a_1, a_2, a_3) : a_i \in \mathbb{R}\}$

Why not study in $\mathbb{R}^3$?

The reason is $\mathbb{R}^3$ is just the set of tripples.
Euclidean 3-space has extra property of

distance & angle

scalar product

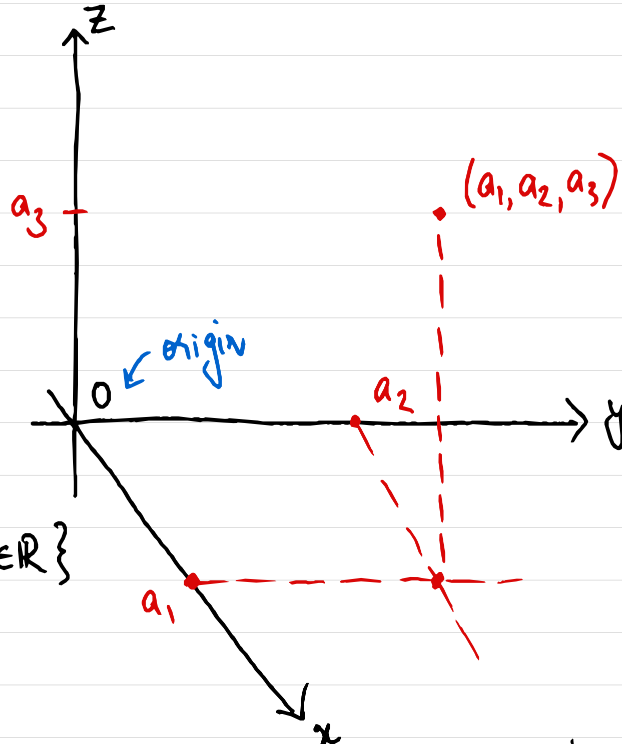
Axes of $\mathbb{E}^3$: Co-ordinate axes

x-axis : $\{(x,0,0) : x \in \mathbb{R}\} \subseteq \mathbb{E}^3$

y-axis : $\{(0,y,0) : y \in \mathbb{R}\} \subseteq \mathbb{E}^3$

z axis : $\{(0,0,z) : z \in \mathbb{R}\} \subseteq \mathbb{E}^3$

Orientate axes according to right hand rule:



$\mathbb{E}^3 = \{(a_1, a_2, a_3) : a_i \in \mathbb{R}\}$

$O := (0,0,0) \in \mathbb{E}^3$

↑ origin

Any point in $\mathbb{E}^3$ of form (x, y, z)

↳ cartesian co-ordinates

$(x\text{-axis}) \cap (y\text{-axis}) \cap (z\text{-axis})$

$\parallel$

$\{(0,0,0)\}$

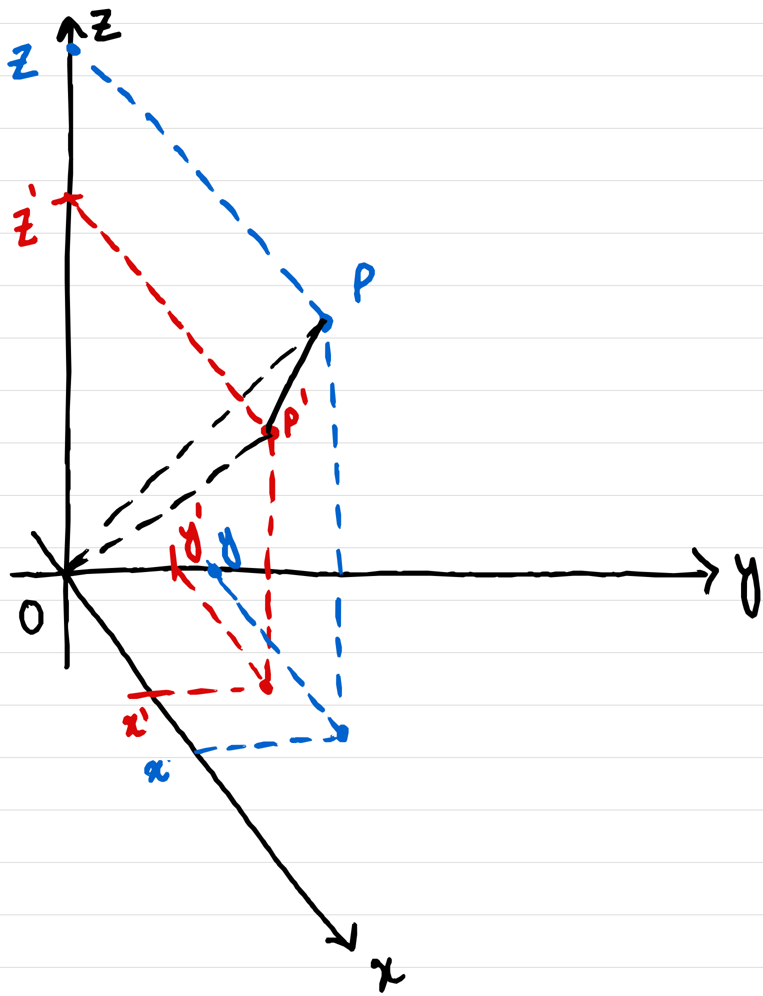
↳ origin

Distance in $\mathbb{E}^3$

Let P and P' be points in $\mathbb{E}^3$, $P, P' \in \mathbb{E}^3$

P has co-ordinates (x, y, z)

P' has co-ordinates (x', y', z')



Need to measure 3 <u>distances</u>:

$$OP, \quad OP', \quad PP'$$

$$\shortparallel \qquad \shortparallel \qquad \shortparallel$$

$$PO \qquad P'O \qquad P'P$$

Define

$$OP = \sqrt{(x^2 + y^2 + z^2)}$$

$$OP' = \sqrt{((x')^2 + (y')^2 + (z')^2}$$

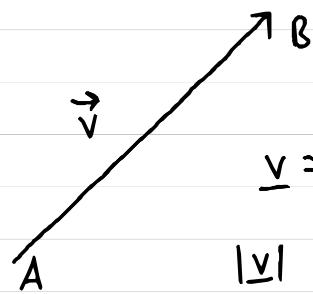
$$PP' = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

↳ basically distance formula

Definition of Vectors:

Vector is a quantity that has both magnitude (size) and direction.

Geometrically a vector can be thought of a directed line segment from one point to another.

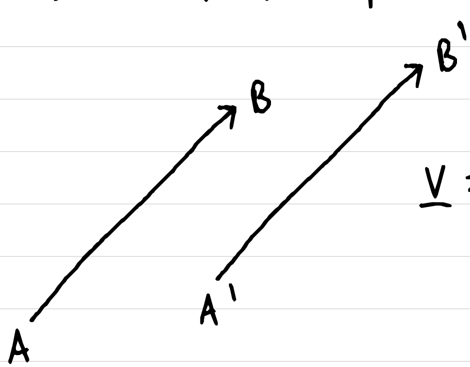


$\underline{v} = \overrightarrow{AB}$

magnitude

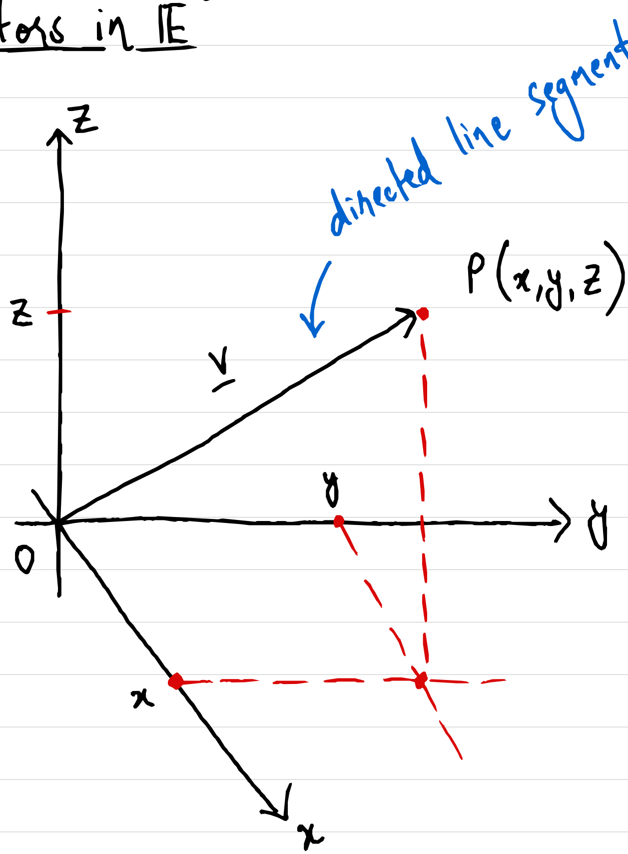
$|\underline{v}|$ is the distance from A to B.

Two vectors are the same if they have the same size and direction even if they are drawn in different parts of space



$\underline{v} = \overrightarrow{AB} = \overrightarrow{A'B'}$

Vectors in $\mathbb{E}^3$



$\vec{OP} = \underline{v}$

In particular any vector can be thought of starting at the origin and going to a certain point P.

So vectors in $\mathbb{E}^3$ is denoted by directed line segments.

Here

$\underline{v} = \overrightarrow{OP}$ → the vector starting at O and ending at P.

initial point

↳ terminal point

Caution:

$\overrightarrow{OP}$ is different to OP

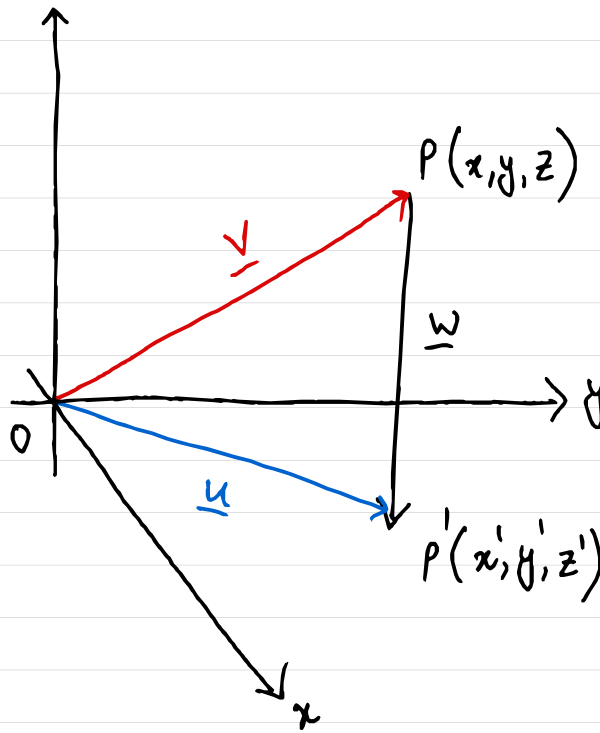
$\overrightarrow{OP}$ is the directed line segment / vector

OP is the distance $|\underline{v}|$ between O and P

Any vector with <u>initial point</u> as <u>origin</u> is called the <u>position vector</u>

Here $\underline{v} = \overrightarrow{OP}$ is a position vector

Notations for vectors:



$\underline{v} = \overrightarrow{OP} \quad \underline{u} = \overrightarrow{OP'} \quad \underline{w} = \overrightarrow{PP'}$

$\underline{v}' = \overrightarrow{PO} \neq \overrightarrow{OP} = \underline{v} \Rightarrow \underline{v}' \neq \underline{v}$

$\underline{w}' = \overrightarrow{P'P} \neq \overrightarrow{PP'} = \underline{w} \Rightarrow \underline{w}' \neq w$

$\underline{u}' = \overrightarrow{OP'} \neq \overrightarrow{P'O} = u' \Rightarrow \underline{u}' \neq u$

due to different directions

Alternative notation: when initial and terminal point are given in co-ordinate form:

$$\vec{OP} = (x, y, z)$$

↳ called component form.

If we write $\underline{v} = (x, y, z)$

↳ we mean that $\underline{v}$ is a vector from origin to point (x, y, z)

So here

$$\vec{OP} = \underline{v} = (x, y, z)$$

Common notation: Column form:

$$\underline{v} \ (\text{or } \vec{OP}) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Magnitude/size/Length of a vector

magnitude of $\underline{v} = (x, y, z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is

denoted by $|\underline{v}|$

$$|\underline{v}| = \sqrt{x^2 + y^2 + \bar{z}}$$

↳ similar to the distance formula from origin

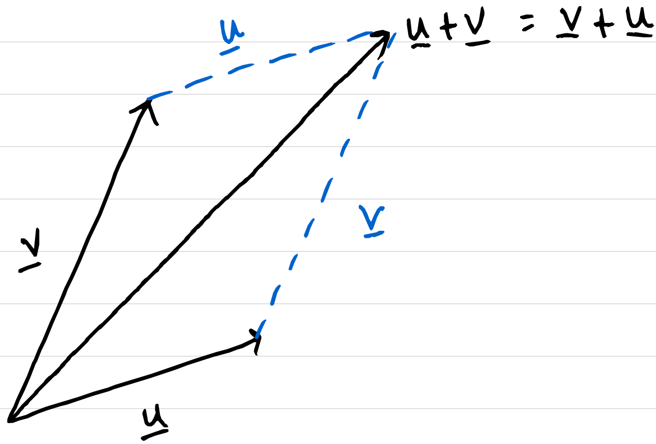
and

$$|\underline{v}| \in [0, \infty) \quad \text{or} \quad |\underline{v}| \in \mathbb{R}_0^*$$

Vector Algebra:

Vector Addition:

Two vectors of same dimension can be added.



Geometrically it is clear that you get the same effect as travelling along $\underline{v}$ and then along $\underline{u}$

Think of $\underline{u} + \bar{v}$ as the "<u>net effect</u>" of going from travelling along $\bar{u}$ and then along $\bar{v}$

Geometrically it is clear that vector addition is commutative,

$$\underline{u}+\underline{v} = \underline{v}+\underline{u}$$

We also give an algebraic proof in $\mathbb{E}^3$

Thm: vector addition is commutative

proof: let $\underline{v}, \underline{u} \in \mathbb{E}^3$

let $\underline{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad \underline{u} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$

$$\underline{v} + \underline{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$= \begin{bmatrix} x+x' \\ y+y' \\ z+z' \end{bmatrix}$$

$$= \begin{bmatrix} x'+x \\ y'+y \\ z'+z \end{bmatrix}$$ commutativity in $\mathbb{R}$

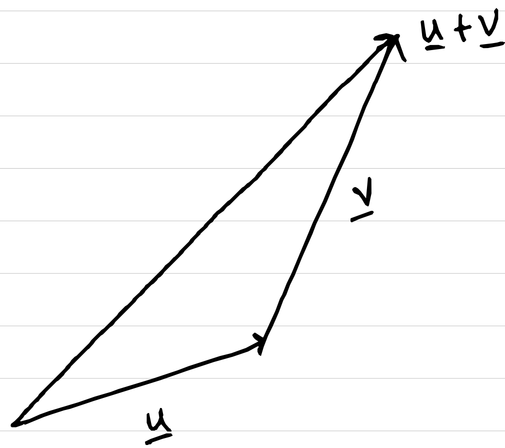
$$= \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} + \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \underline{u} + \underline{v}$$

$$\Rightarrow \quad \underline{v} + \underline{u} = \underline{u} + \underline{v}$$

$$\blacksquare$$

Think of $\underline{u} + \bar{v}$ as the "<u>net effect</u>" of going from travelling along $\bar{u}$ and then along $\bar{v}$



Vector addition in $\mathbb{E}^3$

Let $\underline{u}, \underline{v} \in \mathbb{E}^3$ then

Let $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $\quad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ Then

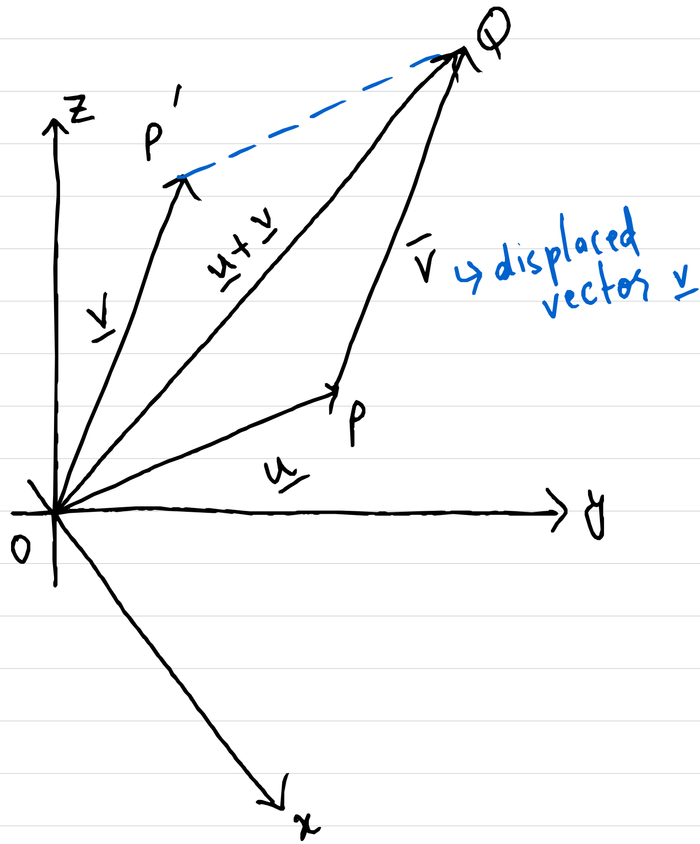
$\underline{u} = (u_1, u_2, u_3)$ $\quad \underline{v} = (v_1, v_2, v_3)$

$$\underline{u} + \underline{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$$

and $\underline{u} + \underline{v} \in \mathbb{E}^3$

Vectors have a very nice geometric interpretation on $\mathbb{E}^3$

(Given on next page)



OPQP' is a parallelogram and

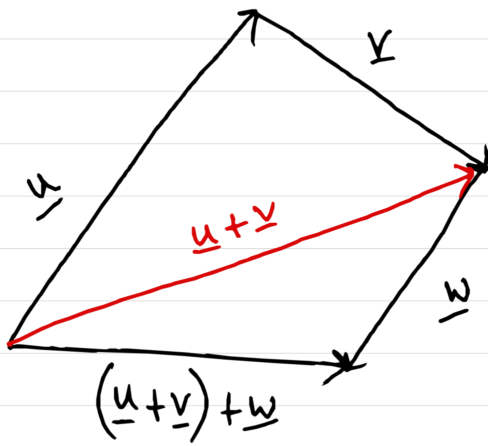
$$\underline{u} + \underline{v} = \overrightarrow{OQ} = \bar{v} + \underline{u}$$

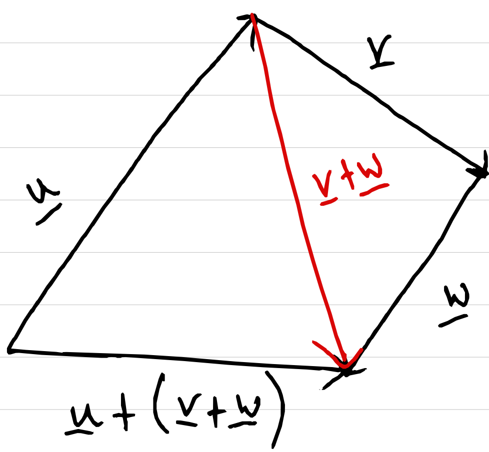
is the diagonal of parallelogram

Thm: Vector addition is associative. let $\underline{u}, \underline{v}, \underline{w} \in \mathbb{E}^3$

$$\underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$$

Geometric proof:





Algebraic proof:

let $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $\quad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ $\quad \underline{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$

$$\underline{u} + (\underline{v} + \underline{w}) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \right)$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} (v_1 + w_1) \\ (v_2 + w_2) \\ (v_3 + w_3) \end{bmatrix}$$

$$= \begin{bmatrix} u_1 + (v_1 + w_1) \\ u_2 + (v_2 + w_2) \\ u_3 + (v_3 + w_3) \end{bmatrix}$$

$$= \begin{bmatrix} (u_1 + v_1) + w_1 \\ (u_2 + v_2) + w_2 \\ (u_3 + v_3) + w_3 \end{bmatrix} \quad \left(\begin{matrix} \text{associativity in} \\ \mathbb{R} \end{matrix} \right)$$

$$= \begin{bmatrix} (u_1 + v_1) \\ (u_2 + v_2) \\ (u_3 + v_3) \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

$$= \left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

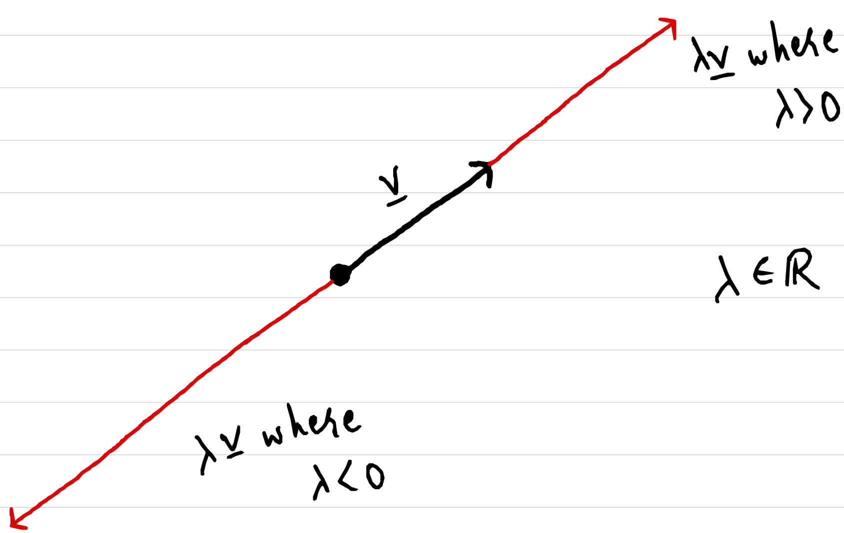
$$= (\underline{u} + \underline{v}) + \underline{w}$$

$$\Rightarrow \quad \underline{u} + (\underline{v} + \underline{w}) = (\underline{u} + \underline{v}) + \underline{w}$$

$\blacksquare$

Scalar Multiplication:

Vectors can be multiplied by scalars (real numbers) to get another vector.



- The new vector is in the same direction as the old vector if $\lambda > 0$

- The new vector is in the opposite direction as the old vector if $\lambda < 0$

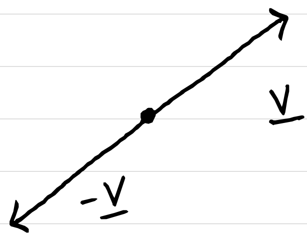
- The new vector has magnitude

$$|\lambda \underline{v}| = |\lambda| \cdot |\underline{v}|$$

So
- if $|\lambda| > 1$, it is longer, $|\lambda \underline{v}| > |\underline{v}|$
- if $|\lambda| < 1$, it is shorter $|\lambda \underline{v}| < |\underline{v}|$
- if $|\lambda| = 1$, then $|\lambda \underline{v}| = |\underline{v}|$

In particular, if $\lambda = -1$, $\lambda \underline{v} = -\underline{v}$.

$-\underline{v}$ is the vector with same magnitude but opposite direction to $\underline{v}$



Scalar Multiplication in $\mathbb{E}^3$

Let $\lambda \in \mathbb{R}$, called scalar.

Let $\underline{v} \in \mathbb{E}^3$ where $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

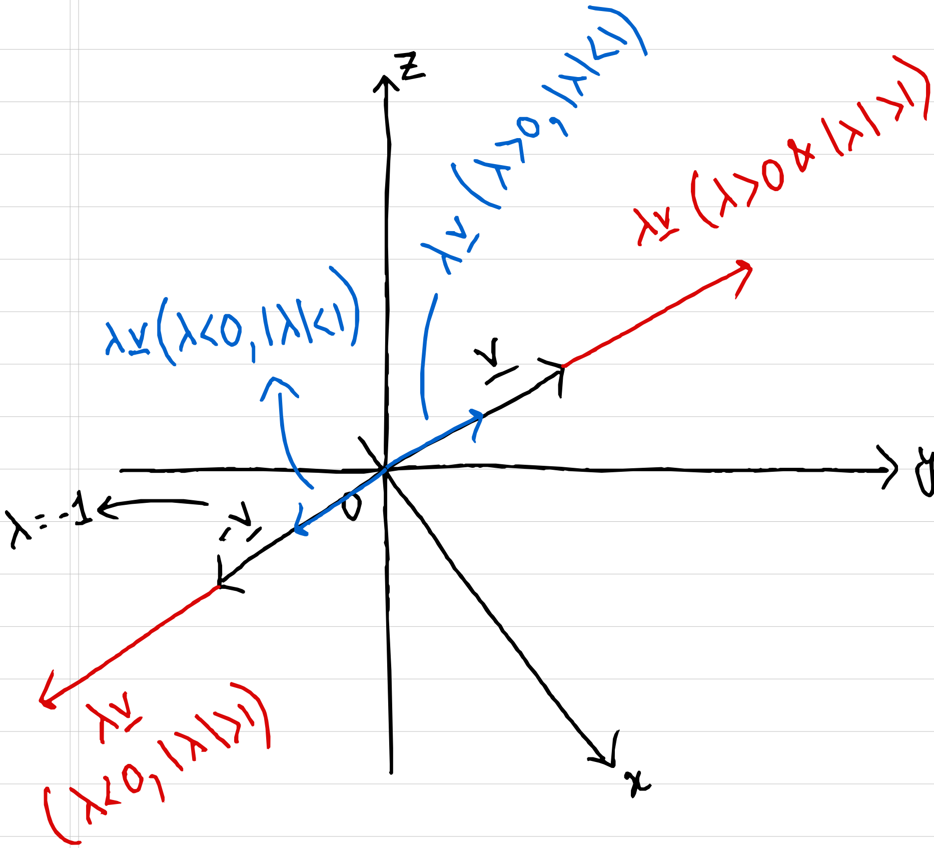
$$\lambda \underline{v} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{bmatrix}$$

Effect of scalar multiplication depends on

sign of λ and magnitude of λ

sign of λ
- $\to$ $\lambda > 0$ then $\lambda \underline{v}$ is in <u>same</u> direction of $\underline{v}$
- $\to$ $\lambda < 0$ then $\lambda \underline{v}$ is in (exact) opposite direction

magnitude
- $\to$ $|\lambda| < 1$ then $|\lambda \underline{v}| < |\underline{v}|$
- $\to$ $|\lambda| > 1$ then $|\lambda v| > |\underline{v}|$

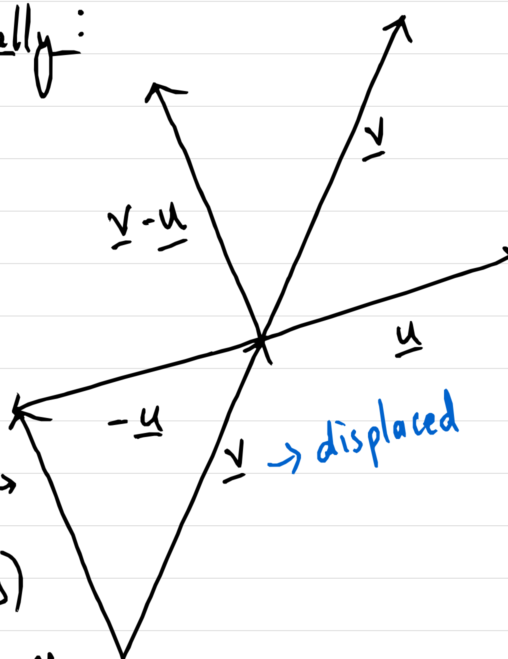


Vector subtraction:

In particular we can subtract a vector from say $\underline{v}$ by adding a vector multiplied by -1. i.e. to subtract $\underline{u}$ from $\underline{v}$

$$\underline{v} - \underline{u} = \underline{v} + -1 . \underline{u}$$

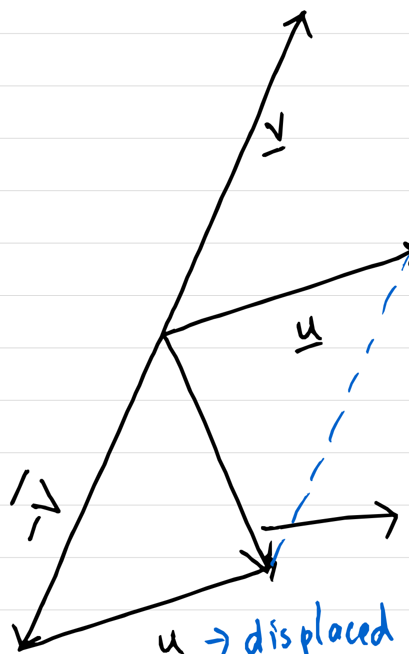
Geometrically:



$\underline{v} - \underline{u} \rightarrow$

$= \underline{v} + (-\underline{u})$

$= \underline{v} + -1 . \underline{u}$



$\underline{u} - \underline{v}$

$= \underline{u} + (-\underline{v})$

$= \underline{u} + -1 . (\underline{v})$

Vector subtraction in $\mathbb{E}^3$

Let $\lambda = 1 \in \mathbb{R}$, $\underline{u}, \underline{v} \in \mathbb{E}^3$,

let $\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ Then

$$\underline{v} - \underline{u} = \underline{v} + (-1 \cdot \underline{u})$$

$$= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + -1 . \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} -u_1 \\ -u_2 \\ -u_3 \end{bmatrix}$$

$$= \begin{bmatrix} v_1 - u_1 \\ v_2 - u_2 \\ v_3 - u_3 \end{bmatrix}$$

$\Rightarrow$

$$\underline{v} - \underline{u} = \begin{bmatrix} v_1 - u_1 \\ v_2 - u_2 \\ v_3 - u_3 \end{bmatrix}$$

Properties of scalar multiplication

Lemma: Let $\lambda, \mu \in \mathbb{R}$, $\underline{u}, \underline{v} \in \mathbb{E}^3$

- (Scalar multiplication is distributive over vector addition.)

$$\boxed{\lambda(\underline{u}+\underline{v}) = \lambda\underline{u} + \lambda\underline{v}}$$

- (Scalar multiplication is associative):

$$\boxed{(\lambda\mu)\underline{v} = \lambda(\mu\underline{v})}$$

- (Scalar addition is distributive):

$$\boxed{(\lambda+\mu)\underline{v} = \lambda\underline{v} + \mu\underline{v}}$$

- (1 is the multiplicative identity):

$$\boxed{1\underline{v} = \underline{v}}$$

- $$\boxed{\underbrace{0}_{\in\mathbb{R}} \cdot \underline{v} = \underbrace{\underline{0}}_{0\text{ vector}}}$$

- (0 vector $\underline{0}$ is the additive identity):

$$\boxed{\underline{v} + \underline{0} = \underline{v}}$$

Co-ordinates, Unit Vectors & Standard basis vectors

Unit Vectors

$\underline{u} \in \mathbb{E}^3$ is a unit vector if and only if

$$|\underline{u}| = 1$$

So a unit vector is vector whose length is one.

Standard basis vectors:

Also called standard unit vectors is a collection of 3 vectors denoted by

$\{\underline{i}\ \underline{j}\ \underline{k}\}$ where

$$\underline{i} = (1,0,0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{k} = (0,0,1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\underline{j} = (0,1,0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$|\underline{i}| = |\underline{j}| = |\underline{k}| = 1$$

Any vector can be represented by the **standard basis vectors.**

Suppose you are given a point P

$$P = (x, y, z)$$

The vector $\overrightarrow{OP}$ can be represented as

called $\underline{k}$ component

$$\overrightarrow{OP} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\underline{i} + y\underline{j} + z\underline{k}$$

called $\underline{j}$ component

called the $\underline{i}$ component

The standard unit vectors are meant to point along the direction of standard axis in cartesian space.

Vector Algebra with standard basis vectors

Let $\underline{u}, \underline{v} \in \mathbb{E}^3$, $\lambda \in \mathbb{R}$

- $\underline{u} \pm \underline{v} = \begin{bmatrix} u_1 \pm v_1 \\ u_2 \pm v_2 \\ u_3 \pm v_3 \end{bmatrix} = (u_1 \pm v_1)\underline{i} + (u_2 \pm v_2)\underline{j} + (u_3 \pm v_3)\underline{k}$

- $\lambda(\underline{u}) = \begin{bmatrix} \lambda u_1 \\ \lambda u_2 \\ \lambda u_3 \end{bmatrix} = (\lambda u_1)\underline{i} + (\lambda u_2)\underline{j} + (\lambda u_3)\underline{k}$

The zero vector is

$$\underline{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

Parallel Vectors

If two vectors are $\underline{u}, \underline{v} \in \mathbb{E}^3$ are parallell / in the same direction as each other then

$$\underline{v} = \lambda \underline{u} \quad \text{or} \quad \underline{u} = \lambda \underline{v}$$

for $\lambda \in \mathbb{R}$

__Example__ Point P has co-ordinates - $P=(1,-2,3)$

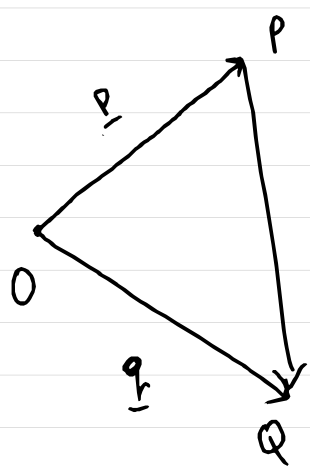
Point Q has co-ordinates - $Q=(-4,-5,6)$

Position vectors are

$$\overrightarrow{OP} = \underline{p} = \underline{i} - 2\underline{j} + 3\underline{k} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$$\overrightarrow{OQ} = \underline{q} = -4\underline{i} - 5\underline{j} + 6\underline{k} = \begin{bmatrix} -4 \\ -5 \\ 6 \end{bmatrix}$$

The displacement vector $\overrightarrow{PQ}$ is $\underline{q} + (-\underline{p})$



(Not accurate diagram, drawn for intuition)

$$\overrightarrow{PQ} = \underset{\sim}{q} - \underset{\sim}{p}$$

$$= \begin{bmatrix} -4 \\ -5 \\ 6 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -4-1 \\ -5-(-2) \\ 6-3 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ 3 \end{bmatrix}$$

Therefore $\overrightarrow{PQ} = -5\hat{i} - 3\hat{j} + 3\hat{k}$

distance between P and Q is $|\overrightarrow{PQ}|$

$$|\overrightarrow{PQ}| = \sqrt{(-5)^2 + (-3)^2 + (3)^2}$$

$$= \sqrt{43}$$

$$\Rightarrow |\overrightarrow{PQ}| = \sqrt{43}$$

Scalar/Dot Product

Scalar or dot product is a function denoted by '$\cdot$' $\hookrightarrow$ (binary operator symbol)

$$\cdot : \mathbb{E}^3 \times \mathbb{E}^3 \to \mathbb{R}$$

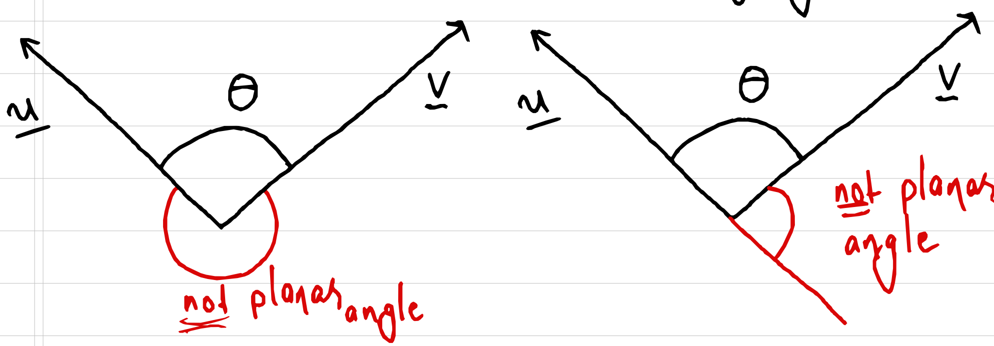
output of $\underline{u}, \underline{v} \notin \mathbb{E}^3$
$\underline{u} \cdot \underline{v} \in \mathbb{R}$ (scalar)

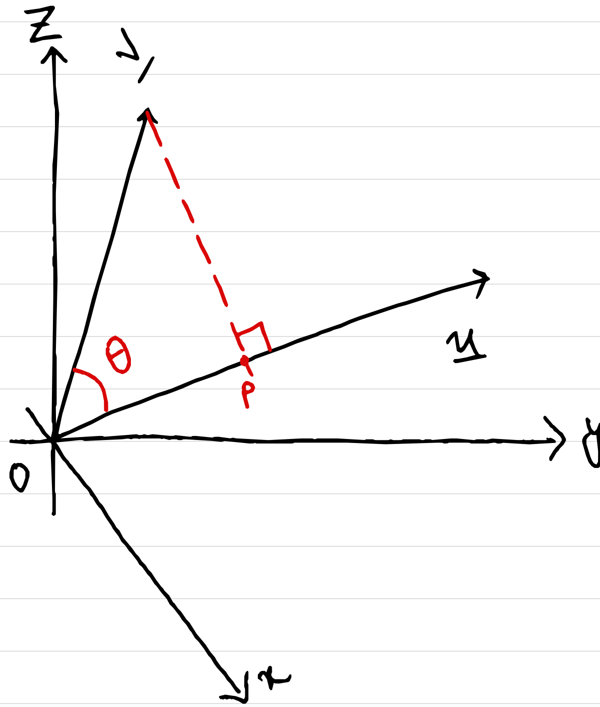
Let $\underline{u}, \underline{v} \in \mathbb{E}^3$. let $\theta \in \mathbb{R}$ be their planar angle (that is the angle between them in the plane spanned by $\underline{u}$ and $\underline{v}$).

Choose the planar angle θ s.t

$$0 \leq \theta \leq \pi$$

and it goes <u>between two vectors going outwards</u>.





Vector $\overrightarrow{OP}$ is in the same direction as $\underline{u}$
So

$$\overrightarrow{OP} = \lambda \underline{u}$$

By trigonometry of $\cos\theta$,

$$\cos\theta = \frac{|\overrightarrow{OP}|}{|\underline{v}|} \Rightarrow |\overrightarrow{OP}| = \underbrace{|\underline{v}|\cos\theta}$$

↳ called "projection" of $\underline{v}$ onto $\underline{u}$
↳ $\text{proj}_{\underline{v}}(\underline{u})$

Now,

$|\vec{OP}| = |\underline{v}|\cos\theta$ as seen before.

Since $\vec{OP}$ is projection of $\underline{v}$ onto $\underline{u}$:

$$\vec{OP} = \lambda \underline{u} = |\underline{v}|\cos\theta\, \hat{\underline{u}}$$

$\hat{\underline{u}}$ is a unit vector in direction of $\underline{u}$ defined by

$$\hat{\underline{u}} = \frac{1}{|\underline{u}|} \cdot \underline{u}$$

In general any vector $\underline{v}$ can be represented by

$$\boxed{\underline{v} = |\underline{v}|\hat{\underline{v}}}$$

where $\hat{\underline{v}}$ is the unit vector in direction of $\underline{v}$

$$\boxed{\hat{\underline{v}} = \frac{1}{|\underline{v}|} \cdot \underline{v}}$$

So

$$\vec{OP} = |\underline{v}|.\cos\theta.\hat{\underline{u}}$$

$$= \underbrace{\frac{|\underline{v}|.\cos\theta}{|\underline{u}|}}_{\lambda}.\,\underline{u} \qquad (\vec{OP} = \lambda \underline{u})$$

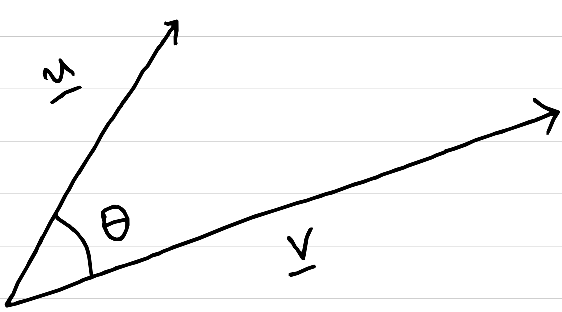
So definition of scalar product is as follows:

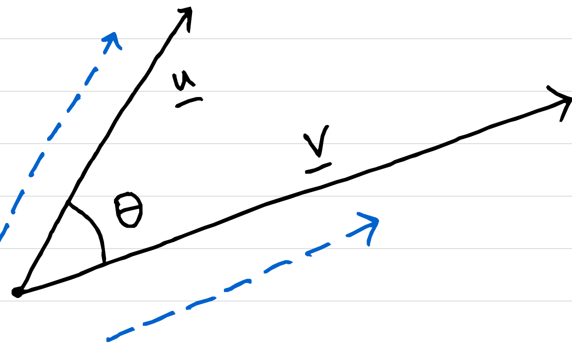
Defn: Let $\underline{u}, \underline{v} \in \mathbb{E}^3$, $\theta \in \mathbb{R}$. The scalar product $\underline{u}.\underline{v} \in \mathbb{R}$ is

Then

$$\boxed{\underline{u} \cdot \underline{v} = |\underline{u}| \times |\underline{v}| \cos\theta} \rightarrow \text{scalar product}$$

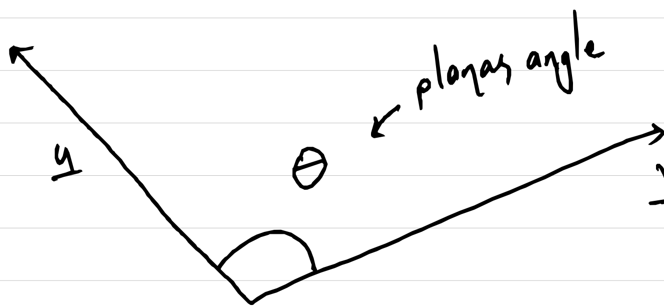
where θ is the planar angle between $\underline{u}$ and $\underline{v}$.





Two vectors that don't lie on the same line always lie on a plane.

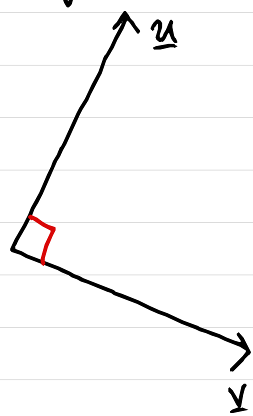
The planar angle is the angle that lies in between the two vectors, when they start from the same point and you measure the angle as you "move out" along the two vectors



By convention $\theta \in [0, \pi] \Rightarrow 0 \leq \theta \leq \pi$

Remember: θ is in radians NOT degrees

Orthogonal Vectors:



Here

$$\underline{u}.\underline{v} = |u| \times |v| \cos\theta$$

$$= |u| \times |v| \cos\frac{\pi}{2}$$

$$= 0$$

Therefore

vectors $\underline{u}$ and $\underline{v}$ are orthogonal/perpendicular

$$\Longleftrightarrow$$

$$\underline{u}.\underline{v} = 0$$

Key observation:

The set of standard basis vectors $\{\underline{i}, \underline{j}, \underline{k}\}$ by construction are orthogonal since they lie on co-ordinate axes.

So $\boxed{\underline{i}\cdot\underline{j} = \underline{j}\cdot\underline{k} = \underline{k}.\underline{i} = 0}$

Properties of Scalar product

- The dot product is **commutative**

$$\boxed{\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}}$$

proof: Since the planar angle between $\underline{u}$ and $\underline{v}$ is the same as planar angle b/w $\underline{v}$ and $\underline{u}$,

$$\underline{u} \cdot \underline{v} = |u| \cdot |v| \cos\theta = |v| \cdot |u| \cos\theta = \underline{v} \cdot \underline{u}$$

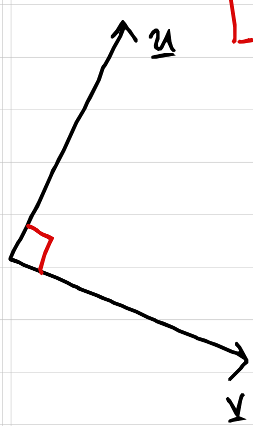
Therefore

$$\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$$

$\blacksquare$

- Two non-zero vectors are **Orthogonal** if and only if

$$\boxed{\underline{u} \cdot \underline{v} = 0}$$



Here

$$\underline{u} \cdot \underline{v} = |u| \times |v| \cos\theta$$

$$= |u| \times |v| \cos\frac{\pi}{2}$$

$$= 0$$

$\blacksquare$

- We have $\boxed{\underline{u} \cdot \underline{u} = |u|^2}$

proof: This follows that if two vectors are parallel then $\theta = 0$ and $\cos(\theta) = 1$

$$\underline{u} \cdot \underline{u} = |u| \cdot |u| \cos\theta = |u|^2 \cdot \cos 0 = |u|^2$$

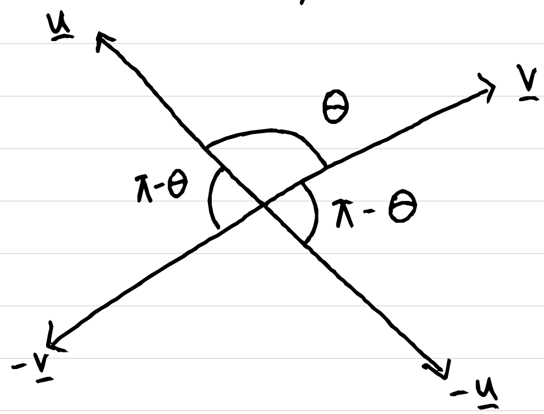
$\blacksquare$

- Let $\underline{u}, \underline{v} \in \mathbb{E}^3$, $\lambda \in \mathbb{R}$

$$\boxed{(\lambda \underline{u}) \cdot \underline{v} = \lambda(\underline{u} \cdot \underline{v}) = \underline{u} \cdot (\lambda \underline{v})}$$

proof: $\underline{u} \cdot (\lambda \underline{v}) = \begin{cases} |\underline{u}||\underline{v}||\lambda| \cos\theta & \text{if } \lambda \geq 0 \\ |\underline{u}||v||\lambda| \cos(\pi - \theta) & \text{if } \lambda < 0 \end{cases}$

$$= (\lambda \underline{u}) \cdot \underline{v}$$



Case 1: $\lambda > 0$

$$(\lambda \underline{u}) \cdot \underline{v} = |\lambda \underline{u}| . |\underline{v}| \cos\theta = |\lambda| |\underline{u}| |\underline{v}| \cos\theta$$

since $\lambda \underline{u}$ is in same direction as $\underline{u}$ as $\lambda > 0$

Now $|\lambda| |\underline{u}| |\underline{v}| \cos\theta = \lambda |\underline{v}| |\underline{u}| \cos\theta = \lambda (\underline{u} \cdot \underline{v})$

since $\lambda > 0$

$$\Rightarrow (\lambda \underline{u}) \cdot \underline{v} = \lambda (\underline{u} \cdot \underline{v})$$

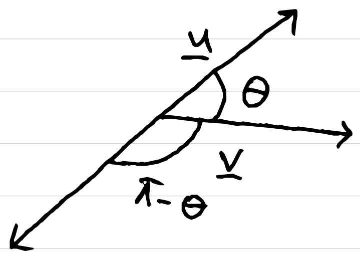
Also $|\lambda| |\underline{u}| |\underline{v}| \cos\theta = |\underline{u}| |\lambda| |\underline{v}| \cos\theta$

$$= |\underline{u}| |\lambda \underline{v}| \cos\theta$$

$$= \underline{u} (\lambda \underline{v}| \cos\theta.$$

Case 2: $\lambda < 0$

If $\lambda < 0$ then λu is in opposite direction as $\underline{u}$, so the planar angle b/w $\lambda \underline{u}$ and $\underline{v}$ is $\pi - \theta$



Hence $(\lambda \underline{u}) . \underline{v} = |\lambda \underline{u}| \, |\underline{v}| \cos(\pi - \theta)$

$= |\lambda| |\underline{u}| |\underline{v}| \cos(\pi - \theta)$

$= - |\lambda| |u| |v| \cos\theta$ $\left(\cos(\pi - \theta) = -\cos\theta\right)$

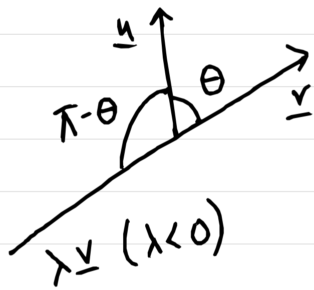
$= -(-\lambda) |u| |v| \cos\theta$ (since $\lambda < 0$)

$= \lambda |u| |v| \cos\theta$

$= \lambda (\underline{u} . \underline{v})$ by defn.

$\Rightarrow (\lambda \underline{u}) . \underline{v} = \lambda (\underline{u} . \underline{v})$

But then also



$$
\begin{aligned}
\underline{u} \cdot (\lambda \underline{v}) &= |\underline{u}| \cdot |\lambda \underline{v}| \cos(\pi - \theta) \\
&= |\underline{u}| \cdot |\lambda| \cdot |\underline{v}| \cos(\pi - \theta) \\
&= |\lambda| \cdot |\underline{u}| \cdot |\underline{v}| \cdot \cos(\pi - \theta) \\
&= -|\lambda| \cdot |\underline{u}| \cdot |\underline{v}| \cdot \cos\theta \qquad (\cos(\pi - \theta) = -\cos\theta) \\
&= -(-\lambda \cdot |u| \cdot |\underline{v}| \cdot \cos\theta) \qquad (\text{as } \lambda < 0) \\
&= \lambda \cdot |\underline{u}| \cdot |\underline{v}| \cos\theta \\
&= \lambda(\underline{u} \cdot \underline{v}) \qquad \text{by defn}
\end{aligned}
$$

$$\Rightarrow \underline{u} \cdot (\lambda \underline{v}) = \lambda(\underline{u} \cdot \underline{v})$$

Case 3: $\lambda = 0$

If $\lambda = 0$, then $(\underline{u}.\underline{v}) = 0$ and

$$|0.\underline{u}| = |0.\underline{v}| = 0$$

So

$$(0.\underline{u}).\underline{v} = \underline{u}.(0.\underline{v}) = 0 = 0.(\underline{u}.\underline{v})$$

$\blacksquare$

- As a corollary of the previous property for any non zero $\underline{v}$

$$\frac{1}{|\underline{v}|}\underline{v} = \frac{\underline{v}}{\sqrt{\underline{v}.\underline{v}}}$$

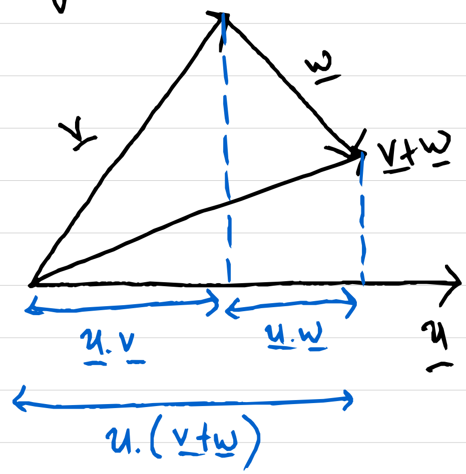
has length one

i.e. $\left|\frac{\underline{v}}{|\underline{v}|}\right| = 1$

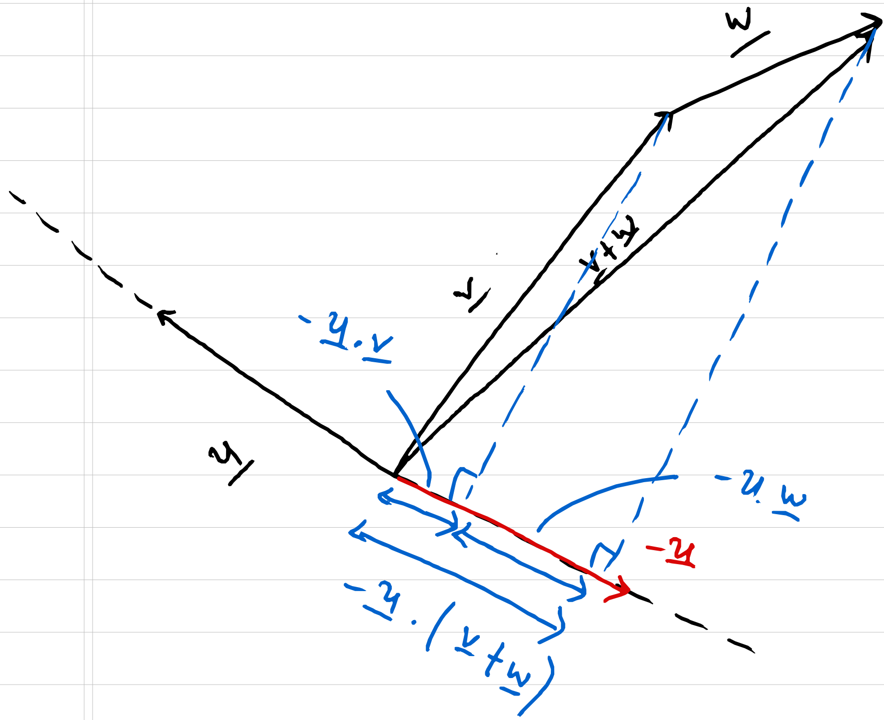
- The dot product is **distributive** over vector addition.

$$\boxed{\underline{u}.(\underline{v}+\underline{w}) = \underline{u}.\underline{v} + \underline{u}.\underline{w}}$$

proof (non-rigorous picture proof):



Note that $\underline{u}.\underline{w}$ can be negative so some of these lengths may be subtracted.



<u>Simple corollary</u>:-

Since $\underline{u}\cdot\underline{u} = |\underline{u}|^2$,

$$\boxed{\underline{i}\cdot\underline{i} = \underline{j}\cdot\underline{j} = \underline{k}\cdot\underline{k} = 1}$$

as $|\underline{i}|^2 = 1$, $|\underline{j}|^2 = 1$, $|\underline{k}|^2 = 1$

Co-ordinate version of dot product

Evaluating $\underline{u} \cdot \underline{v}$

Take the standard basis vectors $\{\underline{i}, \underline{j}, \underline{k}\}$ in $\mathbb{E}^3$

Then there exists (u_1, u_2, u_3) and $(v_1, v_2, v_3) \in \mathbb{R}^3$ such that

$$\underline{u} = u_1 \underline{i} + u_2 \underline{j} + u_3 \underline{k} \quad \text{and}$$

$$\underline{v} = v_1 \underline{i} + v_2 \underline{j} + v_3 \underline{k} \qquad \in \mathbb{E}^3$$

[can also write

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \text{and} \quad \underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$]

Then

$$\boxed{\underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + u_3 v_3} \quad \in \mathbb{R}$$

↳ formula depends on using $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$

$$\underline{u} \cdot \underline{v} = \sum_{i=1}^{3} u_i v_i$$

proof: $\underline{u} \cdot \underline{v} = (u_1 \underline{i} + u_2 \underline{j} + u_3 \underline{k}) \cdot (v_1 \hat{\underline{i}} + v_2 \underline{j} + v_3 \underline{k})$

$$= u_1 \underline{i} (v_1 \underline{i} + v_2 \underline{j} + v_3 \underline{k}) +$$
$$u_2 \underline{j} (v_1 \underline{i} + v_2 \underline{j} + v_3 \underline{k}) +$$
$$u_3 \underline{k} (v_1 \underline{i} + v_2 \underline{j} + v_3 \underline{k}) +$$

(by distributive law of dot product)

$$= u_1 v_1 \underbrace{\underline{i} \cdot \underline{i}}_{1} + u_1 v_2 \overbrace{\underline{i} \cdot \underline{j}}^{0} + u_3 v_3 \overbrace{\underline{j} \cdot \underline{k}}^{0} +$$
$$u_2 v_1 \underline{j} \cdot \underline{i} + u_2 v_2 \underbrace{\underline{j} \cdot \underline{j}}_{1} + u_2 v_3 \underbrace{\underline{j} \cdot \underline{k}}_{0} +$$
$$u_3 v_1 \underbrace{\underline{k} \cdot \underline{i}}_{0} + u_3 v_2 \underbrace{\underline{k} \cdot \underline{j}}_{0} + u_3 v_3 \underbrace{\underline{k} \cdot \underline{k}}_{1}$$

(by distributive law of dot product)

$$= u_1 v_1 + 0 + 0 +$$
$$0 + u_2 v_2 + 0 +$$
$$0 + 0 + u_3 v_3$$

$$\Rightarrow \underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$\blacksquare$

Using the dot product formula to calculate angles

We can use the dot product formula using co-ordinates to calculate planar angle θ.

let $\underline{u}, \underline{v} \in \mathbb{E}^3$, $\theta \in [0, \pi]$, $\underline{u}, \underline{v} \neq \underline{0}$

$$\underline{u} = u_1\underline{i} + u_2\underline{j} + u_3\underline{k} \quad \text{and}$$

$$\underline{v} = v_1\underline{i} + v_2\underline{j} + v_3\underline{k} \qquad \in \mathbb{E}^3$$

$$\underline{u} \cdot \underline{v} = u_1v_1 + u_2v_2 + u_3v_3 = |\underline{u}|.|\underline{v}|.\cos\theta$$

$$\Rightarrow u_1v_1 + u_2v_2 + u_3v_3 = |\underline{u}|.|\underline{v}|\cos\theta$$

$$\Rightarrow \cos\theta = \frac{u_1v_1 + u_2v_2 + u_3v_3}{|\underline{v}|.|\underline{u}|}$$

Therefore

$$\boxed{\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\underline{v}|.|\underline{u}|}\right)}$$

or

$$\boxed{\theta = \cos^{-1}\left(\frac{\sum_{i=1}^{3} u_iv_i}{|\underline{u}|.|\underline{v}|}\right)}$$

<u>Example:</u> Let A B C be points with

$$A = (2,4,2) \quad B = (2,4,1) \quad C = (1,4,2)$$

Angle that lies b/w $\overrightarrow{AB}$ and $\overrightarrow{BC}$

Solution: Let $\overrightarrow{OA} = \underline{a} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ $\quad \overrightarrow{OB} = \underline{b} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ $\quad \overrightarrow{OC} = \underline{c} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \underline{b} - \underline{a} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{BC} = \vec{OC} - \vec{OB}$$

$$= \underline{c} - \underline{b} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{AB} \cdot \vec{BC} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= (0)(-1) + (0)(0) + (-1)(1)$$

$$= -1$$

$$\Rightarrow \vec{AB} \cdot \vec{BC} = -1$$

$$\theta = \cos^{-1}\left(\frac{\vec{AB} \cdot \vec{BC}}{|\vec{AB}| \cdot |\vec{BC}|}\right)$$

$$|\vec{AB}| = \sqrt{0^2+0^2+(-1)^2} = 1$$

$$|\vec{BC}| \sqrt{(-1)^2+0^2+(1)^2} = \sqrt{2}$$

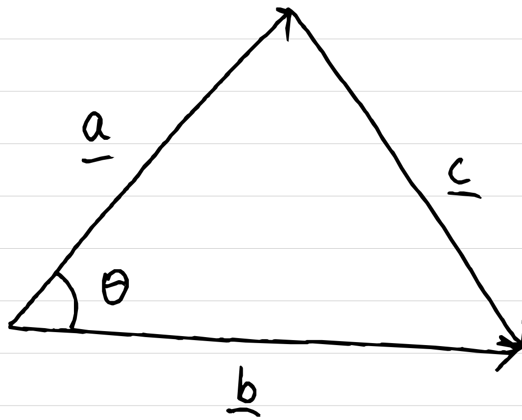
So

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{2}.1}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$$

$$\Rightarrow \theta = \frac{3\pi}{4}$$

Example (Cosine rule):

The dot product enables a quick proof of cosine rule



As $\underline{a}+\underline{c}=\underline{b}$, we have

$$\begin{aligned}\underline{c}\cdot\underline{c} &= (\underline{b}-\underline{a})\cdot(\underline{b}-\underline{a})\\ &= (b-a)\cdot b-(b-a)\cdot a\\ &= \underline{b}\cdot\underline{b}+a\cdot a-2\underline{a}\cdot\underline{b}\end{aligned}$$

$$\Rightarrow \underline{c}\cdot\underline{c}=\underline{b}\cdot\underline{b}+\underline{a}\cdot\underline{a}-2\underline{a}\cdot\underline{b}$$

$$\Rightarrow |\underline{c}|^2=|\underline{b}|^2+|\underline{a}|^2-2|\underline{a}||\underline{b}|\cos\theta$$

$\blacksquare$

Basis for $\mathbb{E}^3$

Canonical basis is $\{\underline{i}, \underline{j}, \underline{k}\}$

But there nothing special about $\{\underline{i}, \underline{j}, \underline{k}\}$ and can use a different basis.

In some examples, it is better to use a different basis than the standard $\{\underline{i}, \underline{j}, \underline{k}\}$

Defn: A basis is a collection of 3 distinct vectors

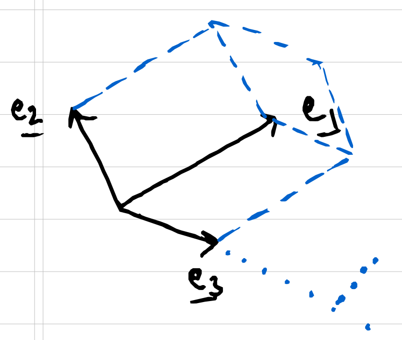
$$\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$$

such that the following properties hold

i) $\underline{e}_1, \underline{e}_2$ and $\underline{e}_3$ span $\mathbb{E}^3$

↳ span means:
Take any $\underline{x} \in \mathbb{E}^3$.
Then $\exists\, x_1, x_2, x_3 \in \mathbb{R}$ s.t

$$\underline{x} = x_1 \underline{e}_1 + x_2 \underline{e}_2 + x_3 \underline{e}_3$$



vectors $\underline{e}_1, \underline{e}_2, \underline{e}_3$ form a parallelopiped

(ii) A given set of $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ forms a basis if the vectors are **linearly independant**

Defn $\underline{e}_1, \underline{e}_2, \underline{e}_3$ are **linearly independant** if and only if

$$\boxed{\lambda_1 \underline{e}_1 + \lambda_2 \underline{e}_2 + \lambda_3 \underline{e}_3 = 0}$$

only when $\lambda_1 = \lambda_2 = \lambda_3 = 0$

The key is that you can have infinitely different basis for $\mathbb{E}^3$

When you fix a basis, $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

$$\underline{x} = \underline{e}_1 x_1 + \underline{e}_2 x_2 + \underline{e}_3 x_3$$

(x_1, x_2, x_3) are the co-ordinates of x with respect to the basis $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

So with basis $\{\underline{i}, \underline{j}, \underline{k}\}$, we get usual coordinates x, y, z for any vector.

Basis $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ are orthogonal

$$\Updownarrow$$

$$\underline{e}_1 \cdot \underline{e}_2 = \underline{e}_1 \cdot \underline{e}_3 = \underline{e}_3 \cdot \underline{e}_1 = 0$$

So canonical basis is orthogonal.

Further

A basis is <u>orthonormal</u>

$$\Updownarrow$$

it is orthogonal and $|e_i| = 1$ for $i = 1, 2, 3$

The scalar product formula

The value is unchanged

$$\underline{a} \cdot \underline{b} = \sum_{i=1}^{n} a_i b_i$$

changing basis changes this sum

depends on the basis of $\{\underline{i}, \underline{j}, \underline{k}\}$

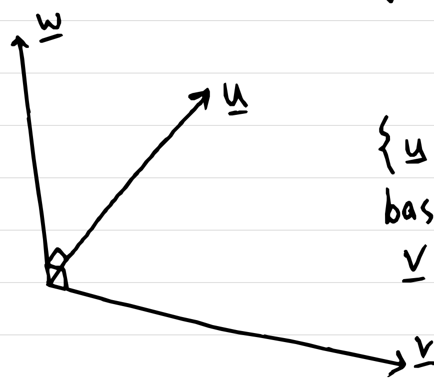
but formula for summation is for scalar product. As no matter basis, the vector remains same and so does the angle between them

Vector (or cross) product

Vector or cross product is a function denoted as '$\times$'

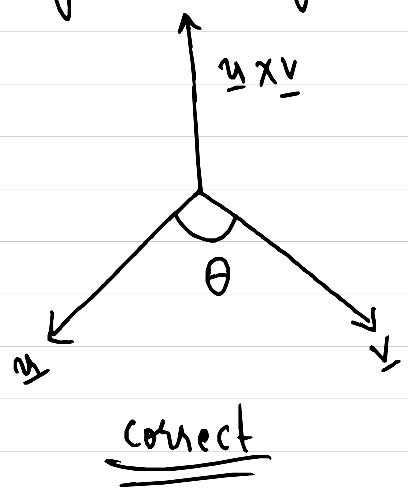
$$\times : \mathbb{E}^3 \times \mathbb{E}^3 \rightarrow \mathbb{E}^3$$ → output is a vector $\underline{u} \times \underline{v} \in \mathbb{E}^3$

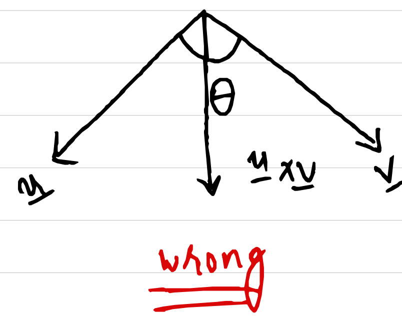
Motivation: Given two (non-zero) vectors $\underline{u}$ and $\underline{v}$ construct a vector $\underline{w}$ which is orthogonal to <u>both</u> $\underline{u}$ and $\underline{v}$



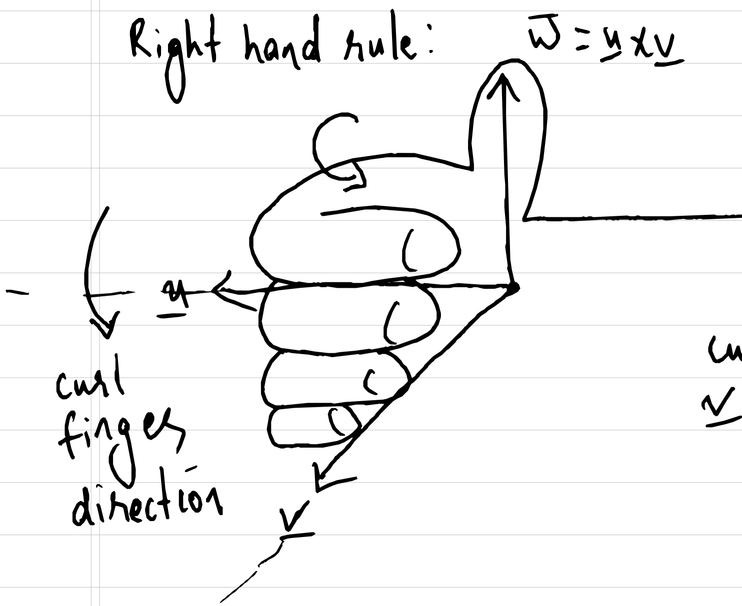
$\{\underline{u}, \underline{v}, \underline{w}\}$ will be a basis for $\mathbb{E}$ if $\underline{u}$ and $\underline{v}$ are not collinear

Defn: The cross product between two three dimensional vectors $\underline{u}$ and $\underline{v}$ is a vector with length $|u||v|\sin\theta$ where θ is the planar angle b/w $\underline{u}$ and $\underline{v}$ and in the direction orthogonal to both $\underline{u}$ and $\underline{v}$ according to the right hand rule.





Right hand rule:



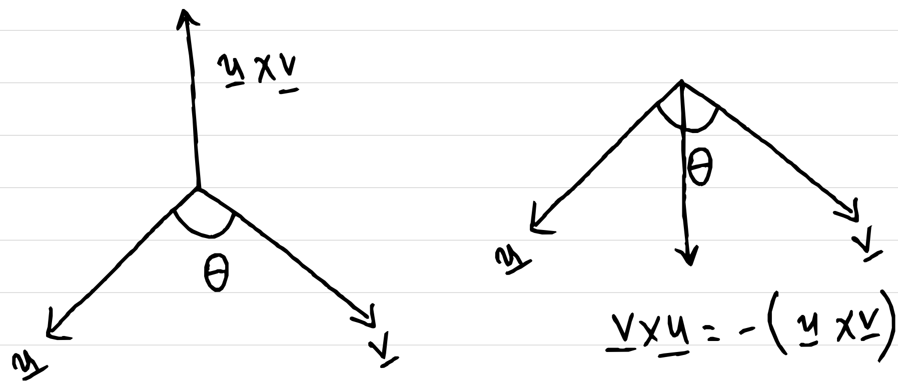
place right edge hand in direction of first vector $\underline{u}$

curl figers toward $\underline{v}$ to get $\underline{u} \times \underline{v}$

Some consequences of geometric defn:

1) Note that right hand rule implies

$$\boxed{\underline{u} \times \underline{v} = -(\underline{v} \times \underline{u})}$$



2) If $\underline{v} = \lambda \underline{u}$ then

$$\theta = \begin{cases} 0 & \text{if } \lambda > 0 \\ \pi & \text{if } \lambda < 0. \end{cases}$$

Since $\sin(0) = \sin(\pi) = 0$, so

$$|\underline{v} \times \underline{u}| = |\underline{v}||\underline{u}| \sin\theta = 0$$

$$\Rightarrow \underline{v} \times \underline{u} = \underline{0} \qquad \text{(zero vector)}$$

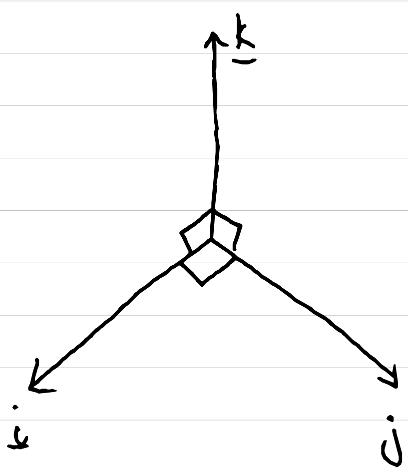
The way to think about this is that if $\underline{u}$ and $\underline{v}$ are parallel (they point in same or opposite directions), then the cross product is $\underline{0}$.

3) By property 2:

$$\boxed{\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = \underline{0}}$$

4) Three vectors $\underline{i}$ $\underline{j}$ $\underline{k}$ are chosen with the right hand rule, are of length one and are at mutual right angles.

↳ angle is $\pi/2$



Since $\sin(\pi/2) = 1$

$$\boxed{\begin{aligned} \underline{i} \times \underline{j} &= \underline{k} \\ \underline{j} \times \underline{k} &= \underline{i} \\ \underline{k} \times \underline{i} &= \underline{j} \end{aligned}}$$

5) For any real $\lambda \in \mathbb{R}$

$$\boxed{\lambda(\underline{u}) \times \underline{v} = \lambda(\underline{u} \times \underline{v}) = \underline{u} \times \lambda(\underline{v})}$$

_proof: <u>Case $\lambda > 0$:</u>

If $\lambda > 0$ then • $\lambda \underline{u}$ has same direction as $\underline{u}$

• $\lambda \underline{v}$ has same direction as $\underline{v}$

• $\lambda(\underline{u} \times \underline{v})$ has same direction as $\underline{u} \times \underline{v}$

$\lambda \underline{u}$ has same direction as $\underline{v}$, So $\lambda \underline{u} \times \underline{v}$ has same direction as $\underline{u} \times \underline{v}$

Now $\lambda(\underline{u} \times \underline{v})$ has same direction as $\underline{u} \times \underline{v}$.
Therefore

$\lambda \underline{u} \times \underline{v}$ has same direction as $\lambda(\underline{u} \times \underline{v})$

Let planar angle be θ b/w $\underline{u}$ and $\underline{v}$. Since $\lambda \underline{u}$ is in same direction as $\underline{u}$. So the planar angle b/w $\underline{u}$ and $\underline{v}$ is the same as $\lambda \underline{u}$ and $\underline{v} = \theta$

The size/magnitude of vector $\lambda \underline{u} \times \underline{v}$ is

$$|\lambda \underline{u} \times \underline{v}| = |\lambda \underline{u}| . |\underline{v}| \sin\theta$$

$$= |\lambda| . |\underline{u}| . |\underline{v}| \sin\theta$$

$$= \lambda . |\underline{u}| \, |\underline{v}| \sin\theta \quad (\text{as } \lambda > 0)$$

$$= \lambda |\underline{u} \times \underline{v}| \quad (\text{by defn of } |\underline{u} \times \underline{v}|)$$

$$= |\lambda| \, |\underline{u} \times \underline{v}|$$

$$= |\lambda (\underline{u} \times \underline{v})|$$

So $\lambda(\underline{u}) \times \underline{v}$ and $\lambda(\underline{u} \times \underline{v})$ has same length and direction so

$$\lambda(\underline{u} \times \underline{v}) = \lambda(\underline{u}) \times \underline{v}$$

Very similar argument for

$$\lambda(\underline{u} \times \underline{v}) = \underline{v} \times \lambda(\underline{u})$$

So

$$\lambda(\underline{u}) \times \underline{v} = \lambda(\underline{u} \times \underline{v}) = \underline{u} \times \lambda(\underline{v})$$

If $\lambda < 0$ then • $\lambda \underline{u}$ has opposite direction as $\underline{u}$

• $\lambda \underline{v}$ has opposite direction as $\underline{v}$

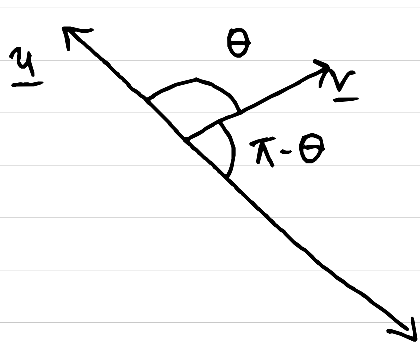
• $\lambda(\underline{u} \times \underline{v})$ has opposite direction as $(\underline{u} \times \underline{v})$

$\lambda \underline{u}$ has opposite direction of $\underline{u}$ so by right hand rule, $\lambda(\underline{u}) \times \underline{v}$ points in opposite direction of $\underline{u} \times \underline{v}$

$\lambda(\underline{u} \times \underline{v})$ points in opposite direction to $\underline{u} \times \underline{v}$

Therefore $\lambda \underline{u} \times \underline{v}$ is in same direction to $\lambda(\underline{u} \times \underline{v})$.

Furthermore if planar angle b/w $\underline{u}$ and $\underline{v}$ is θ then the planar angle b/w $\lambda \underline{u}$ and $\underline{v}$ is $\pi - \theta$, as $\underline{v}$ is in opposite direction to $\underline{u}$.



Since $\sin(\pi - \theta) = \sin\theta$, the size/magnitude/length of $(\lambda)\underline{u} \times \underline{v}$ is

$$|\lambda(\underline{u}) \times \underline{v}| = |\lambda \underline{u}||\underline{v}|\sin\theta$$

$$= |\lambda||\underline{u}||\underline{v}|\sin\theta$$

$$= -\lambda|\underline{u}||\underline{v}|\sin\theta \quad \text{(as } \lambda < 0)$$

$$= -\lambda|\underline{u} \times \underline{v}| \quad \text{(by defn of } |\underline{u} \times \underline{v}|)$$

$$= |\lambda||\underline{u} \times \underline{v}|$$

$$= |\lambda(\underline{u} \times \underline{v})|$$

So $\lambda(\underline{u}) \times \underline{v}$ and $\lambda(\underline{u} \times \underline{v})$ have direction and magnitude so

$$\lambda(\underline{u}) \times \underline{v} = \lambda(\underline{u} \times \underline{v})$$

Very similar argument for $\underline{u} \times \lambda(\underline{v}) = \lambda(\underline{u} \times \underline{v})$

So

$$\lambda(\underline{u}) \times \underline{v} = \lambda(\underline{u} \times \underline{v}) = \underline{u} \times \lambda(\underline{v})$$

For $\lambda = 0$,

$$\lambda(\underline{u}) \text{ and } \lambda(\underline{v}) \text{ and } \lambda(\underline{u} \times \underline{v}) = \underline{0}$$

So

$$\lambda(\underline{u}) \times \underline{v} = \lambda(\underline{u} \times \underline{v}) = \underline{u} \times \lambda(\underline{v}) = \underline{0}$$

$\blacksquare$

6) Distributive law for cross product

$$\boxed{\underline{u} \times (\underline{v} + \underline{w}) = \underline{u} \times \underline{v} + \underline{u} \times \underline{w}}$$

proof: ommited $\blacksquare$

Co-ordinate Version of Cross Product.

Method 1: Assume all co-ordinates relative to standard basis.

Let $\underline{u}, \underline{v} \in \mathbb{E}^3$,

$$\underline{u} = (u_1, u_2, u_3) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\underline{v} = (v_1, v_2, v_3) = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$\underline{w}$ is a function of $\underline{u}$ and $\underline{v}$

Aim: define $\underline{w}(\underline{u}, \underline{v})$ such that

$$\underline{w} \cdot \underline{u} = 0 \quad \text{and} \quad \underline{w} \cdot \underline{v} = 0$$

and $\underline{w} \neq \underline{0}$

Let $\underline{w} \in \mathbb{E}^3$ and

$$\underline{w} = (w_1, w_2, w_3) = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

So

$\underline{w} \cdot \underline{u} = 0 \Rightarrow w_1 u_1 + w_2 u_2 + w_3 u_3 = 0$ $(*1)$

$\underline{w} \cdot \underline{v} = 0 \Rightarrow w_1 v_1 + w_2 v_2 + w_3 v_3 = 0$ $(*2)$

Multiplying $(*1)$ by v_3

$v_3 (w_1 u_1 + w_2 u_2 + w_3 u_3) = 0$

$\Rightarrow w_1 u_1 v_3 + v_3 w_2 u_2 + v_3 u_3 w_3 = 0$ $(*3)$

Multiplying $(*2)$ by u_3

$u_3 (w_1 v_1 + w_2 v_2 + w_3 v_3) = 0$

$\Rightarrow w_1 v_1 u_3 + w_2 v_2 u_3 + w_3 v_3 u_3 = 0$ $(*4)$

Subtracting $(*3)$ and $(*4)$

$$w_1 u_1 v_3 + v_3 w_2 u_2 + \cancel{v_3 u_3 w_3} - (w_1 v_1 u_3 + w_2 v_2 u_3 + \cancel{w_3 v_3} u_3) = 0$$

$\Rightarrow w_1 u_1 v_3 + v_3 w_2 u_2 - w_1 v_1 u_3 - w_2 v_2 u_3 = 0$

$\Rightarrow w_1 (u_1 v_3 - v_1 u_3) + w_2 (v_3 u_2 - u_3 v_2) = 0$

$\Rightarrow w_1(u_1v_3 - v_1u_3) = -w_2(u_2v_3 - u_3v_2)$

Let $w_1 = u_2v_3 - u_3v_2$ and

$-w_2 = u_1v_3 - v_1u_3 \Rightarrow w_2 = v_1u_3 - u_1v_3$

Substite values back into equations (*1) or (*2)

$w_1u_1 + w_2u_2 + w_3u_3 = 0$

$\Rightarrow (u_2v_3 - u_3v_2)u_1 + (v_1u_3 - u_1v_3)u_2 + w_3u_3 = 0$

$\Rightarrow u_1\cancel{u_2}v_3 - u_1u_3v_2 + v_1u_2u_3 - \cancel{u_1u_2}v_3 + w_3u_3 = 0$

$\Rightarrow v_1u_2u_3 - u_1u_3v_2 + w_3u_3 = 0$

$\Rightarrow u_3(v_1u_2 - u_1v_2 + w_3) = 0$

Therefore let

$v_1u_2 - u_1v_2 + w_3 = 0 \Rightarrow w_3 = u_1v_2 - u_2v_1$

So we have

$w_1 = u_2 v_3 - u_3 v_2$, $w_2 = v_1 u_3 - u_1 v_3$ and $w_3 = u_1 v_2 - u_2 v_1$

Define the vector or cross product of

$\underline{u} = (u_1, u_2, u_3)$ and $\underline{v} = (v_1, v_2, v_3)$ to be

$$\boxed{\underline{u} \times \underline{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ v_1 u_3 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}}$$

All co-ordinates are respect to basis $\{\underline{i}, \underline{j}, \underline{k}\}$

Method 2:

Lemma: Let $\underline{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$, $\underline{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$
$\underline{u}, \underline{v} \in \mathbb{E}^3$

Then

$$\boxed{\underline{u} \times \underline{v} = (u_2v_3 - u_3v_2)\hat{i} + (v_1u_3 - u_1v_3)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}}$$

proof: $\underline{u} \times \underline{v} = (u_1\hat{i} + u_2\hat{j} + u_3\hat{k}) \times (v_1\hat{i} + v_2\hat{j} + v_3\hat{k})$

$$= \cancel{u_1v_1\hat{i}\times\hat{i}} + u_1v_2\hat{i}\times\hat{j} + u_1v_3\hat{i}\times\hat{k}$$

$$+ u_2v_1\hat{j}\times\hat{i} + \cancel{u_2v_2\hat{j}\times\hat{j}} + u_2v_3\hat{j}\times\hat{k}$$

$$+ u_3v_1\hat{k}\times\hat{i} + u_3v_2\hat{k}\times\hat{j} + \cancel{u_3v_3\hat{k}\times\hat{k}}$$

Due to the fact that $\hat{i}\times\hat{i} = 0 = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$
and

$$\hat{i}\times\hat{j} = \hat{k}, \quad \hat{j}\times\hat{k} = \hat{i}, \quad \hat{k}\times\hat{i} = \hat{j}$$

$$\hat{j}\times\hat{i} = -\hat{k} \quad \hat{k}\times\hat{j} = -\hat{i}, \quad \hat{i}\times\hat{k} = -\hat{j}$$

We get

$$\underline{u} \times \underline{v} = (u_2v_3 - u_3v_2)\hat{i} + (v_1u_3 - u_1v_3)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}$$

$\blacksquare$

Showing that $\underline{u}$ is orthogonal to $\underline{u} \times \underline{v}$

$$\underline{u} \cdot (\underline{u} \times \underline{v}) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - v_3 u_1 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

$$= u_1 (u_2 v_3 - u_3 v_2) + u_2 (u_3 v_1 - v_3 u_1) + u_3 (u_1 v_2 - u_2 v_1)$$

$$= \cancel{u_1 u_2 v_3} - \cancel{u_1 u_3 v_2} + \cancel{u_2 u_3 v_1} - \cancel{u_2 v_3 u_1} + \cancel{u_3 u_1 v_2} - \cancel{u_3 u_2 v_1}$$

$$= 0$$

$$\Rightarrow u \cdot (\underline{u} \times \underline{v}) = 0$$

Similarly it can be shown that

$$\underline{v} \cdot (\underline{u} \times \underline{v}) = 0$$

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = u_1 \underline{i} + u_2 \underline{j} + u_3 \underline{k}$$

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \underline{i} + v_2 \underline{j} + v_3 \underline{k}$$

$$\boxed{\underline{u} \times \underline{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - v_3 u_1 \\ u_1 v_2 - u_2 v_1 \end{bmatrix} \Rightarrow \underline{u} \cdot (\underline{u} \times \underline{v}) = 0 \text{ and } \underline{v} \cdot (\underline{u} \times \underline{v}) = 0}$$

$$\boxed{\underline{u} \times \underline{u} = \underline{0} \quad \forall\, \underline{u} \in \mathbb{E}^3}$$

$$\underline{u} \times \underline{u} = \begin{bmatrix} u_2 u_3 - u_3 u_2 \\ u_3 u_1 - u_1 u_3 \\ u_1 u_2 - u_2 u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \underline{0}$$

So in particular

$$\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = 0$$

Useful way of determing vector product using determinants

Vector equation of a line

Suppose that line L is a line passing through point $P_0(x_0, y_0, z_0)$ (has position vector $\underline{p}$) and is parallel to vector $\underline{v} = v_1\underline{i} + v_2\underline{j} + v_3\underline{k}$

Then L is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to $\underline{v}$. Thus

$$P_0P = t\underline{v} \quad \text{for } t \in \mathbb{R}. \text{ (scalar parameter)}$$

The value of t depends on the location of the poin P along the line, and domain of t is $(-\infty, \infty) = \mathbb{R}$.

The expanded form of $t\underline{v}$ is

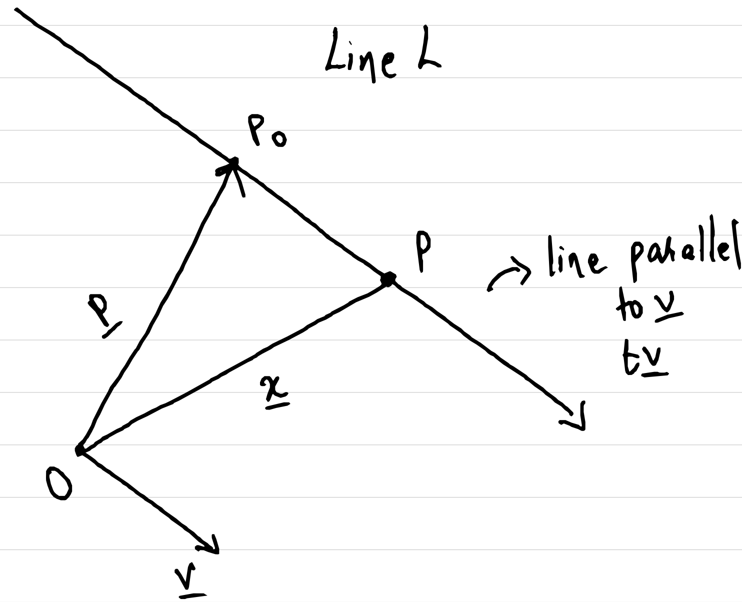
$$\overrightarrow{P_0P} = t\underline{v} \quad \Rightarrow$$

$$(x - x_0)\underline{i} + (y - y_0)\underline{j} + (z - z_0)\underline{k} = t(v_1\underline{i} + v_2\underline{j} + v_3\underline{k})$$

which can be rewritten as

$$x\underline{i} + y\underline{j} + z\underline{k} = x_0\underline{i} + y_0\underline{j} + z_0\underline{k} + t(v_1\underline{i} + v_2\underline{j} + v_3\underline{k})$$

$$\Rightarrow \underline{x} = P_0 + t\underline{v} \Rightarrow \underline{x} = \underline{p} + t\underline{v}$$



$\underline{x} = \underline{p} + t\underline{v}$

The position vector $\underline{x}$ of an arbitrary point $P(x, y, z)$ on the line in terms of $\underline{p}$ and $\underline{v}$

$$\underline{x} = \underline{p} + t\underline{v} \quad \text{for } t \in \mathbb{R}$$

"To get to P, first travel to P_0, then go along t units in the $\underline{v}$ direction."

Example: Find the vector equation of the line going through point $\underline{p} = 3\underline{i} + 4\underline{j}$ in the direction $\underline{v} = 2\underline{i} - 4\underline{j}$

<u>Solution:</u> $\underline{x} = \underline{p} + t\underline{v} = 3\underline{i} + 4\underline{j} + 2t\hat{\imath} - t\underline{j} + 3t\underline{k}$

$$= (3+2t)\underline{i} + (4-t)\underline{j} + 3t\underline{k}$$

Parametric equation of a line

Note that every point $\underline{x}$ on the line can be written as

$$\underline{x} = x\underline{i} + y\underline{j} + z\underline{k} \quad \text{for some } x, y, z \in \mathbb{R}$$

Therefore one can write an equation of a line as

$$\begin{cases} x = x_0 + tv_1 \\ y = y_0 + tv_2 \\ z = z_0 + tv_3 \end{cases} \quad \text{for } t \in \mathbb{R}$$

This is Known as parametric co-ordinate equation of a line
Here x_0, y_0, z_0 are co-ordinates of point P_0.
v_1, v_2, v_3 are co-ordinates of point direction of line.

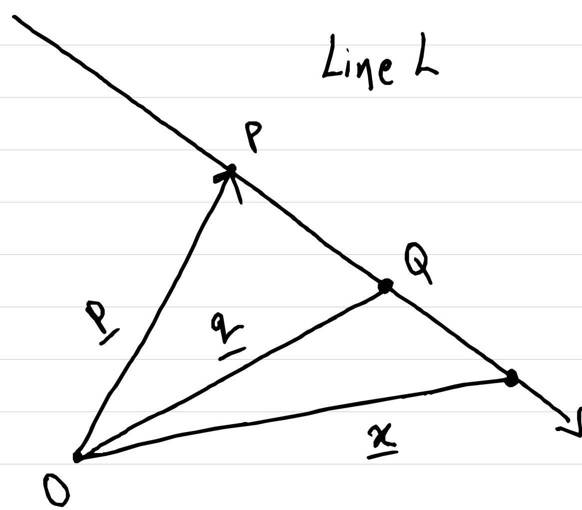
Describing a line going through 2 points

An alternative way to describe a line is to give two points P and Q, the line runs through.

Recall that the direction vector $\overrightarrow{PQ}$ is

$$\overrightarrow{PQ} = \underline{p} - \underline{q}$$

and thus the line is the line that passes through P in direction $\overrightarrow{PQ}$, that is



$$\boxed{\underline{x} = \underline{p} + t(\underline{q} - \underline{p}) \Rightarrow \underline{x} = t\underline{q} + (1-t)\underline{p}}$$

for $t \in \mathbb{R}$

Example: Equation of line going through the points

$$\underline{p} = 7\underline{i} + 2\underline{j} + 6\underline{k}$$

$$\underline{q} = \underline{i} + 5\underline{j} - 3\underline{k}$$

$$\underline{x} = t(\underline{i} + 5\underline{j} - 3\underline{k}) + (1-t)(7\underline{i} + 2\underline{j} + 6\underline{k})$$

and parametric co-ordinate system of the line is

$$x = t + 7(1-t) = 7 - 6t$$

$$y = 5t + 2(1-t) = 2 - 3t$$

$$z = -3t + 6(1-t) = 6 - 9t$$

<u>Showing that a particular point lies on a line:</u>

Example: Show that the point $3\underline{i} + 4\underline{j}$ lies on the same line as in previous example

Solution To show $3i + 4j$ lies on the line, we need to find a particular value of t s.t

$3 = 7 - 6t, \qquad 4 = 2 + 3t \qquad 0 = 9 - 6t$

Note that by solving the 3 simultaneous equations gives $t = 2/3$, so point lies on line

Remark: Recall that in the first example, we found the equation of the line going through $3\underline{i} + 4\underline{j}$ in the direction $\underline{v} = 2\underline{i} - \underline{j} + 3\underline{k}$.
Note that direction of this vector line is $-6\underline{i} + 3\underline{j} - 9\underline{k}$ and that this equals $-3\underline{v}$.
Therefore these two lines are exactly the same.

In otherwords:

To show two vector equations

$$\underline{x} = \underline{p}_1 + t\underline{v}_1 \quad \text{and} \quad \underline{x} = \underline{p}_2 + s\underline{v}_2 \quad \text{for } s, t \in \mathbb{R}$$

are the same line, show that p_1 lies on the second line (that $\exists$ an $s' \in \mathbb{R}$ s.t. $p_1 = p_2 + s'v_2$ and $\underline{v}_1 = \lambda \underline{v}_2$ for some $\lambda \in \mathbb{R}$)

Dividing a line segment into 2 pieces of varying proportions

Dividing a line segment P and Q in the ratio a:b means finding points between P and Q (call it X) such that

$$\frac{|PX|}{|XQ|} = \frac{a}{b}$$

Note that

$$\underline{p} + \underline{t}(\underline{q} - \underline{p}) = t\underline{q} + (1-t)\underline{p} \qquad \text{for } 0 \le t \le 1$$

yields points on the line segment between P and Q. If we put

$$\boxed{t = \frac{a}{a+b}}$$ into the equation,

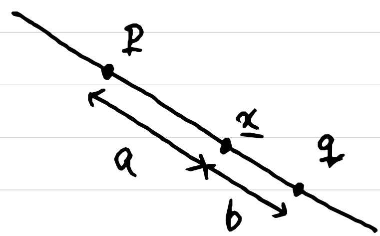
we get

$$\boxed{\underline{x} = \underline{p} + \frac{a}{a+b}(\underline{q} - \underline{p}) = \frac{b}{a+b}\underline{p} + \frac{a}{a+b}\underline{q}}$$

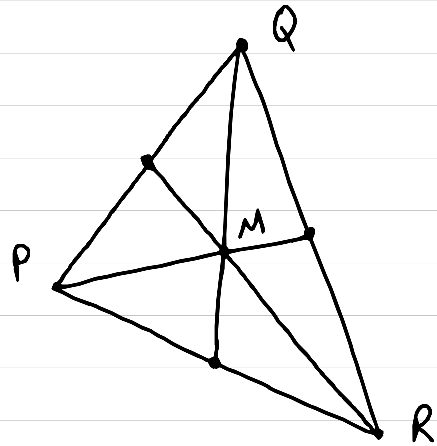
Note that

$$\boxed{\begin{aligned} |\underline{x}-\underline{p}| &= \frac{a}{a+b}\,|\underline{q}-\underline{p}| \quad \text{and} \\ |\underline{x}-\underline{q}| &= \frac{b}{a+b}\,|\underline{p}-\underline{q}| \end{aligned}}$$

That this point divides the line segment in the ratio a:b



Example: Suppose P, Q and R are 3 points. The 3 lines connecting a point to the midpoint of the other 2 points all intersect at a common point, called the centroid of a triangle.



Solution: The midpoint of QR is the point that divides the line segment in the ratio 1:1 which is

$$\frac{1}{2}\underline{q} + \frac{1}{2}\underline{r}$$

Thus the line segment from P to the midpoint of QR is ↳ equation

$$\underline{x}_1 = t_1\underline{p} + (1-t_1)\left(\tfrac{1}{2}\underline{q} + \tfrac{1}{2}\underline{r}\right) \text{ for } 0 \leq t_1 \leq 1$$

↳ for M to exist

Similarly other lines are

$$\underline{x}_2 = t_2\underline{q} + (1-t_2)\left(\tfrac{1}{2}\underline{p} + \tfrac{1}{2}\underline{r}\right) \quad \text{for } 0 \le t_2 \le 1$$

$$\underline{x}_3 = t_3\underline{r} + (1-t_3)\left(\tfrac{1}{2}\underline{p} + \tfrac{1}{2}\underline{q}\right) \quad \text{for } 0 \le t_3 \le 1$$

From this we get simultaneous eqn for t_1, t_2, t_3.

Solving you should get $t_1 = t_2 = t_3 = 1/3$

you get

$$\underline{x}_1 = \underline{x}_2 = \underline{x}_3 = \frac{1}{3}\left(\underline{p} + \underline{q} + \underline{r}\right)$$

which shows there is a point that lies on all three lines, the centroid.

Example: If $\underline{u}$ and $\underline{v}$ are not parallel then only $x, y \in \mathbb{R}$ satisfying

$$x\underline{u} + y\underline{v} = 0$$

is $x = y = 0$

proof: (by contradiction):

Assume the contrary: that is atleast one of x or y is non-zero.

Without loss of generality, assume $x \neq 0$.

Then $xu + y\underline{v} = 0 \Rightarrow x\underline{u} = -y\underline{v}$

x is a non-zero scalar, $x \in \mathbb{R} \setminus \{0\}$

$$x\underline{u} = -y\underline{v} \Rightarrow \underline{u} = \frac{-y}{x}\underline{v}$$

$\Rightarrow$ $\underline{u}$ and $\underline{v}$ are parallel.

This is a contradiction

So $y = 0 = x$ $\blacksquare$

Example: If $\underline{u}$ and $\underline{v}$ are non-colinear and

$$x_1 \underline{u} + y_2 \underline{v} = x_2 \underline{u} + y_2 \underline{v}$$

for some $x_1 x_2 y_1 y_2 \in \mathbb{R}$ then $x_1 = x_2$ & $y_1 = y_2$

proof: Suppose $x_1 \underline{u} + y_1 \underline{v} = x_2 \underline{u} + y_2 \underline{v}$.

Then by previous example

$(x_1 - x_2)\underline{u} + (y_1 - y_2)\underline{v} = 0$ only when

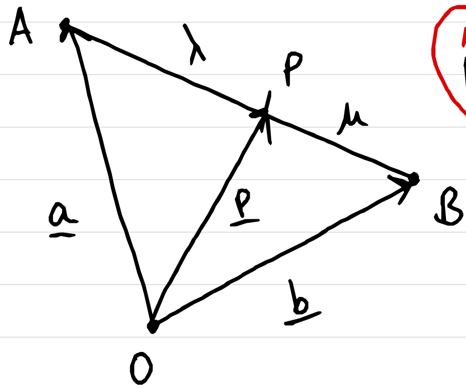
$x_1 - x_2 = 0$ and $y_1 - y_2 = 0$

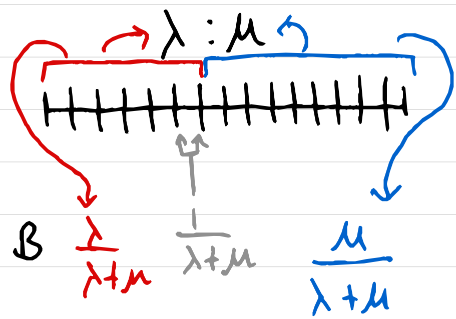
$\Rightarrow$

$x_1 = x_2$ and $y_1 = y_2$.

$\blacksquare$

Example: Take 2 points A and B in $\mathbb{E}^3$. Find position vector of point P that lies on the line joining A and B in the ratio $\lambda : \mu$





Desired position vector $\overrightarrow{OP} = \underline{p}$

Now

$$\overrightarrow{OP} = \underline{p} = \overrightarrow{OA} + \overrightarrow{AP} = \underline{a} + \overrightarrow{AP}$$

$\overrightarrow{AP}$ is parallel to $\overrightarrow{AB} \Rightarrow \overrightarrow{AP} = t(\overrightarrow{AB})$

$\overrightarrow{AB} = \underline{b} - \underline{a} \Rightarrow \overrightarrow{AP} = t(\underline{b} - \underline{a})$

By the diagram above, $t = \dfrac{\lambda}{\mu+\lambda}$

So $\overrightarrow{OP} = \underline{p} = \underline{a} + \dfrac{\lambda}{\mu+\lambda}(\underline{b} - \underline{a})$

So

$$\underline{p} = \underline{a} + \frac{\lambda}{\mu+\lambda}(\underline{b}-\underline{a})$$

$$= \underline{a} + \frac{\lambda \underline{b}}{\mu+\lambda} - \frac{\lambda}{\mu+\lambda}\underline{a}$$

$$= \underline{a}\left(1 - \frac{\lambda}{\mu+\lambda}\right) + \frac{\lambda \underline{b}}{\mu+\lambda}$$

$$= \frac{\mu}{\lambda+\mu}\underline{a} + \frac{\lambda \underline{b}}{\lambda+\mu}$$

$\Rightarrow$

$$\boxed{\underline{p} = \frac{\mu}{\lambda+\mu}\underline{a} + \frac{\lambda}{\lambda+\mu}\underline{b}}$$

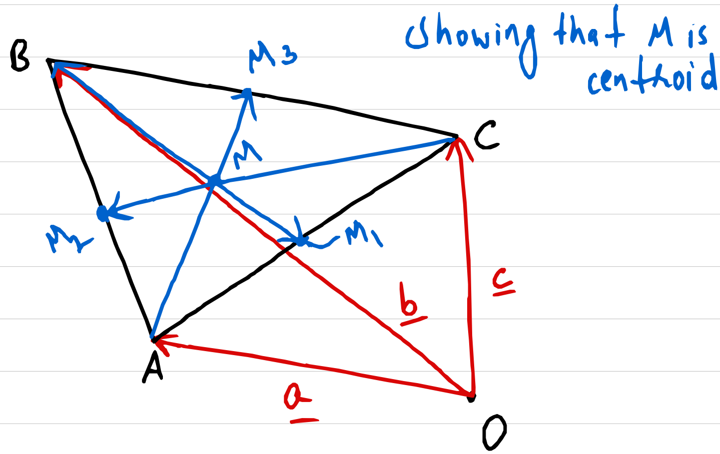
Special case: $1:1$

↖ midpoint

The position vector of midpoint of the line A to B is

$$\boxed{\underline{p} = \frac{1}{2}\underline{a} + \frac{1}{2}\underline{b}}$$

Example: Find the position vector of centroid of a triangle.



To show M is the centroid we need to show that

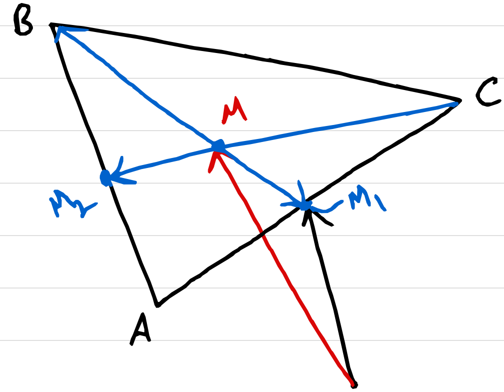
1) M exists, all lines meet at M

2) Show that M is at center of triangle

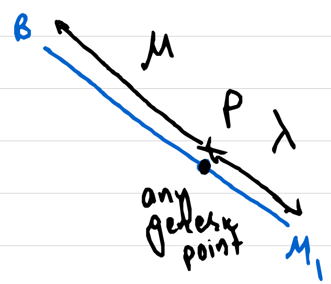
By previous examples:

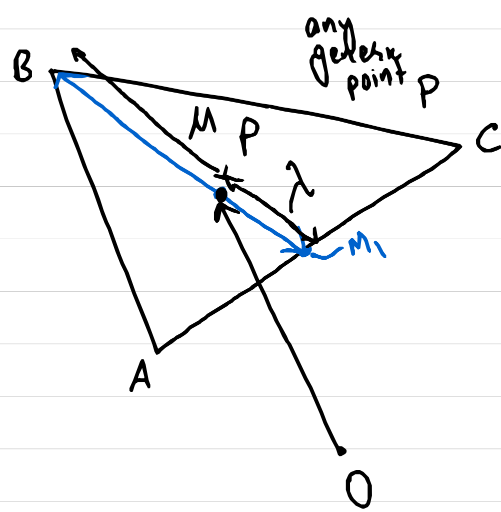
$$OM_1 = \underline{m}_1 = \frac{1}{2}\underline{a} + \frac{1}{2}c \qquad OM_3 = \frac{1}{2}\underline{b} + \frac{1}{2}\underline{c}$$

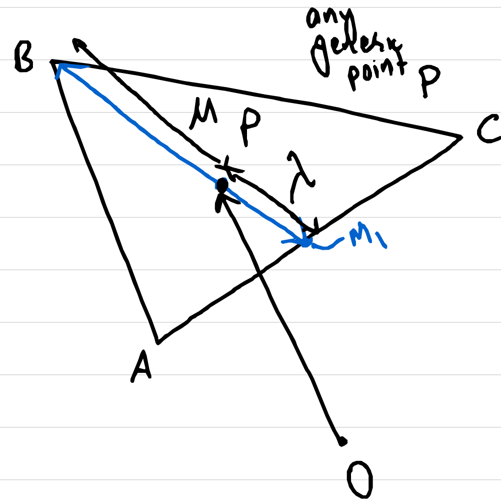
$$OM_2 = m_2 = \frac{1}{2}\underline{a} + \frac{1}{2}\underline{b}$$



Take M_1 and M_2.







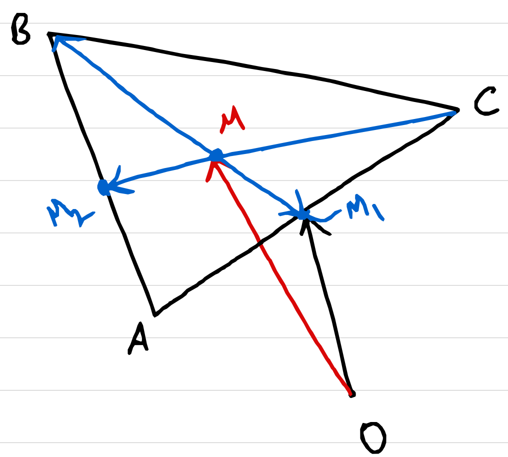
By result proven in previous example

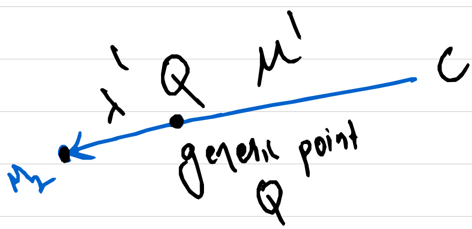
$$\vec{OP} = \frac{\mu}{\lambda+\mu}\left(\vec{OM_1}\right) + \frac{\lambda}{\lambda+\mu}\left(\vec{OB}\right)$$

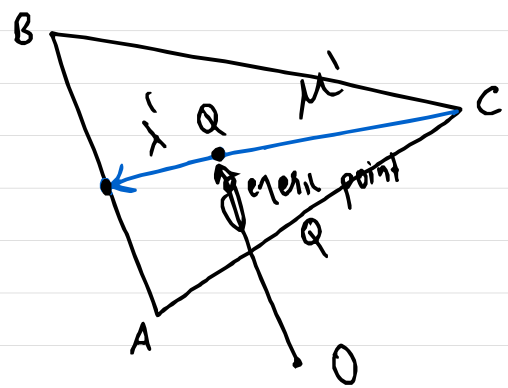
$\Rightarrow$

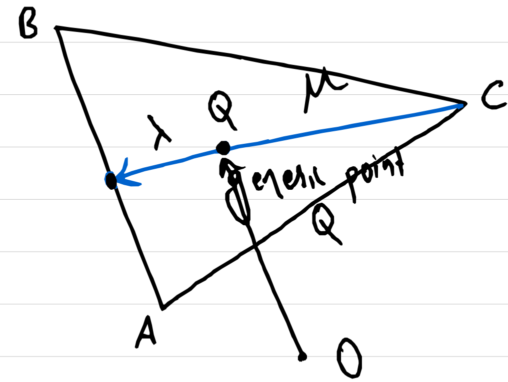
$$\vec{OP} = \frac{\mu}{\lambda+\mu}\left(\frac{1}{2}(\underline{a}+\underline{c})\right) + \frac{\lambda}{\lambda+\mu}\left(\underline{b}\right)$$

Consider the line joining M_2 to C









By result proven in previous example

$$\overrightarrow{OQ} = \frac{\mu'}{\lambda'+\mu'}\left(\overrightarrow{OM_2}\right) + \frac{\lambda'}{\lambda'+\mu'}\left(\overrightarrow{OC}\right)$$

$\Rightarrow$

$$\overrightarrow{OQ} = \frac{\mu'}{\lambda'+\mu'}\left(\frac{1}{2}(\underline{a}+\underline{b})\right) + \frac{\lambda'}{\lambda+\mu'}(\underline{c})$$

showing that $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ intersect

$$\overrightarrow{OP} = \overrightarrow{OQ} \Rightarrow \frac{\mu}{\lambda+\mu}\left(\frac{1}{2}(\underline{a}+\underline{c})\right) + \frac{\lambda}{\lambda+\mu}(\underline{b})$$

$$= \frac{\mu'}{\lambda'+\mu'}\left(\frac{1}{2}(\underline{a}+\underline{b})\right) + \frac{\lambda'}{\lambda+\mu'}(\underline{c})$$

Since $\vec{OP}$ and $\vec{OQ}$ are non collinear
by previous example, comparing co-efficients;

comparing co-efficients of $\underline{a}$,

$$\frac{\mu}{2(\lambda+\mu)} = \frac{\mu'}{2(\lambda'+\mu')} \qquad (*_1)$$

comparing co-efficients of $\underline{b}$

$$\frac{\mu'}{2(\lambda'+\mu')} = \frac{\lambda}{(\lambda+\mu)}$$

$$\Rightarrow \frac{\lambda}{\lambda+\mu} = \frac{\mu}{2(\lambda+\mu)} \qquad (\text{by } (*_1))$$

$$\Rightarrow 2\lambda - \mu = 0$$

$$\Rightarrow \lambda = \frac{\mu}{2}$$

Comparing co-efficients of $\underline{c}$

$$\frac{\mu}{2(\lambda+\mu)} = \frac{\lambda'}{2(\mu'+\lambda')} \Rightarrow \lambda' = \frac{\mu'}{2} \quad \text{(similar calculation)}$$

So as $\lambda = \frac{\mu}{2}$

$$\frac{\lambda}{\lambda+\mu} = \frac{\mu/2}{\mu+\mu/2} = \frac{\mu/2}{\frac{3\mu}{2}} = \frac{1}{3}$$

$$\frac{\mu}{\lambda+\mu} = \frac{\mu}{\frac{\mu}{2}+\mu} = \frac{\mu}{\frac{3\mu}{2}} = \frac{2}{3}$$

As $\lambda' = \frac{\mu'}{2}$

$$\frac{\lambda'}{\lambda'+\mu'} = \frac{\lambda'}{\lambda'+2\lambda'} = \frac{1}{3}$$

$$\frac{\mu'}{\lambda'+\mu'} = \frac{\mu'}{\frac{\mu'}{2}+\mu'} = \frac{\mu'}{\frac{3\mu'}{2}} = \frac{2}{3}$$

The solutions are consistent,
The two lines intersect at

$$\overrightarrow{OP} = \overrightarrow{OQ}$$

$\Rightarrow$

$$\frac{2}{3}\left(\frac{1}{2}\left(\underline{a}+\underline{c}\right)\right) + \frac{1}{3}\underline{b} = \frac{2}{3}\left(\frac{1}{2}\left(\underline{a}+\underline{b}\right)\right) + \frac{1}{3}\underline{c}$$

$\Rightarrow$

$$\frac{1}{3}\left(\underline{a}+\underline{b}+\underline{c}\right) = \frac{1}{3}\left(\underline{a}+\underline{b}+\underline{c}\right)$$

So $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ intersect.

(Finish after notes on planes)

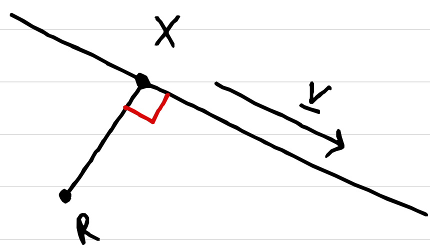
The distance from a point to a line

The distance from a point R to a line is the minimum of $|\underline{x}-\underline{r}|$ where $\underline{x}$ lies on the line. Geometrically it is clear this point will be when the vector from $\underline{r}$ to $\underline{x}$ is orthogonal to the direction.

That is, we are searching for an $\underline{x}$ that lies on the line such that

$$(\underline{x}-\underline{r}).\underline{v} = 0$$

where $\underline{v}$ is a vector parallel to the line.



Recall that a general point on the line can be written as $\underline{p}+t\underline{v}$ for some $t \in \mathbb{R}$ where $\underline{p}$ is the position vector of a point on the line

Thus we wish to find the specific $t \in \mathbb{R}$ such that

$$(\underline{p} + t\underline{v} - \underline{r}) \cdot \underline{v} = 0$$

Expanding out the dot product this is

$$(\underline{p} - \underline{r}) \cdot \underline{v} + t\underline{v} \cdot \underline{v} = 0$$

Therefore

$$t = -\frac{(\underline{p} - \underline{r}) \cdot \underline{v}}{|\underline{v}|^2}$$

replace

$\underline{v} \cdot \underline{v} = |\underline{v}|^2$

Plugging this value of t, in shows the point on the line closes to R has position vector

$$\underline{p} = -\frac{(\underline{p} - \underline{r}) \cdot \underline{v}}{|\underline{v}|^2} \underline{v}$$

The square of the minimal distance is therefore

$$|\underline{x} - \underline{r}|^2 = (\underline{x} - \underline{r}) \cdot (\underline{x} - \underline{r})$$

$$|\underline{x}-\underline{r}|^2 = (\underline{x}-\underline{r}).(\underline{x}-\underline{r})$$

$$= \left(\underline{p}-\underline{r}-\frac{(\underline{p}-\underline{r}).\underline{v}}{|\underline{v}|^2}\underline{v}\right)\cdot\left(\underline{p}-\underline{r}-\frac{(\underline{p}-\underline{r})}{|\underline{v}|^2}\underline{v}\right)$$

$$= |\underline{p}-\underline{r}|^2 - 2\frac{((\underline{p}-\underline{r}).\underline{v})^2}{|\underline{v}^2|} + \left(\frac{(\underline{p}-\underline{r})\cdot\underline{v}}{|\underline{v}|^2}\right)|\underline{v}|^2$$

$$= \frac{|\underline{p}-\underline{r}|^2|\underline{v}|^2 - ((\underline{p}-\underline{r}).\underline{v})^2}{|\underline{v}|^2}$$

Note that if θ is the planar angle between $\underline{p}-\underline{r}$ and the vector $\underline{v}$ then the above equals:

$$\frac{|\underline{p}-\underline{r}|^2|\underline{v}|^2(1-\cos^2\theta)}{|\underline{v}|^2} = \frac{|\underline{p}-\underline{r}|^2|\underline{v}|^2}{|\underline{v}|^2}$$

$$= \frac{|(\underline{p}-\underline{r})\times\underline{v}|^2}{|\underline{v}|^2}$$

Taking the square root shows that the minimal distance of R to a line going through the point P in the direction $\underline{v}$ equals

$$\boxed{|\underline{x}-\underline{r}| = \frac{|(\underline{p}-\underline{r})\times\underline{v}|}{|\underline{v}|}}$$

Vector equations of planes

The vector equation for a plane is similar to that of a line. But it has 2 parameters instead of 1.

Given a point P, with position vector $\underline{p}$ and two vectors $\underline{u}$ and $\underline{v}$ not lying on the same line (that is not colinear), then there is a plane that passes through P parallel to both $\underline{u}$ and $\underline{v}$

The position vector of an arbitrary X in the plane is

$$\boxed{\underline{x} = \underline{p} + s\underline{u} + t\underline{v} \quad \text{for } s,t \in \mathbb{R}}$$

This is known as the plane spanned by $\underline{u}$ and $\underline{v}$ going through $\underline{p}$

Example: The plane going through $\underline{i}+\underline{j}+\underline{k}$ parallel to both $\underline{i}+2\underline{j}+\underline{k}$ and $3\underline{i}-\underline{j}+2\underline{k}$ is

$$\underline{x} = (\underline{i}+\underline{j}+\underline{k}) + s(\underline{i}+2\underline{j}+\underline{k}) + t(3\underline{i}-\underline{j}+2\underline{k})$$
$$= (1+s+3t)\underline{i} + (1+2s-t)\underline{j} + (1+s+2t)\underline{k}$$

Parametric Equation for a plane

The parametric equations for the plane above are

$$\begin{cases} x = 1+s+3t \\ y = 1+2s-t \\ z = 1+s+2t \end{cases}$$

Defining a plane consisting of 3 points

An alternative way to define a plane is to give 3 non-colinear points lying on the plane $\underline{p}, \underline{q}, \underline{r}$

Note that both $\underline{p}-\underline{r}$ and $\underline{q}-\underline{r}$ are vectors parallel to the plane and by definition the point with position $\underline{r}$ lies in the plane, so the vector equation of the plane can be written as

$$\underline{x} = \underline{r} + s(\underline{p}-\underline{r}) + t(\underline{q}-\underline{r})$$
$$= s\underline{p} + t\underline{q} + (1-s-t)\underline{r}$$

So

$$\boxed{\underline{x} = \underline{r} + s(\underline{p}-\underline{r}) + t(\underline{q}-\underline{r}) = s\underline{p} + t\underline{q} + (1-s-t)\underline{r}}$$

This can also be written in a more symmetric form as

$$\boxed{\underline{x} = a\underline{p} + b\underline{q} + c\underline{r}}$$

where $a,b,c \in \mathbb{R}$ such that $a+b+c=1$

Example: Find the plane going through points

$$\underline{p} = \underline{i} + 2\underline{j} + 3\underline{k}, \quad \underline{q} = 3\underline{i} + 2\underline{j} + \underline{k}, \quad \underline{r} = \underline{i} + \underline{j} + \underline{k}$$

Solution: $\underline{p} - \underline{r} = \underline{j} + 2\underline{k}$ $\quad \underline{q} - \underline{r} = 2\underline{i} + \underline{j}$

plane is

$$\underline{x} = (1+2t)\underline{i} + (1+t+s)\underline{j} + (1+2s)\underline{k}$$
$s,t \in \mathbb{R}$

Finding intersection of a line and plane

Unless a line is parallel to the plane, it will intersect the plane at a point.
To find that point, solve 3 simultaneous equations (so find a point on the line (one parameter) that is also a point on the plane (2 parameter)).

Example: Find the point of intersection between the line through $3\underline{i}+6\underline{j}+2\underline{k}$ parallel to $-\underline{i}+2\underline{j}-\underline{k}$ and the plane through $\underline{i}+2\underline{j}+\underline{k}$ and $3\underline{i}-\underline{j}+2\underline{k}$

Solution: First writing equation of line in parametric form

$$\begin{cases} x = 3-t \\ y = 6+2t \\ z = 2-t \end{cases} \qquad t \in \mathbb{R}$$

Secondly writing equation of plane in parametric form:

$$\begin{cases} x = 1+s+3r \\ y = 1+2s-r \\ z = 1+s+2r \end{cases}$$

The common point is when

$$3-t = 1+s+3r$$
$$6+2t = 1+2s-r$$
$$2-t = 1+s+$$